

θ_{13} and The Flavor Ring

Jennifer Kile,^{*} M. Jay Pérez,[†] Pierre Ramond,[‡] Jue Zhang,[§]

*Institute for Fundamental Theory, Department of Physics,
University of Florida, Gainesville, FL 32611, USA*

Abstract

We present the results of a numerical search for the Dirac Yukawa matrices of the Standard Model, consistent with the quark and lepton masses and their mixing angles. We assume a diagonal up-quark matrix, (natural in $\mathbf{Z}_7 \times \mathbf{Z}_3$), Bimaximal or Tri-bimaximal seesaw mixing, and $SU(5)$ unification to relate the down-quark and charged lepton Dirac Yukawa matrices using Georgi-Jarlskog mechanisms. The measured value of θ_{13} requires an asymmetric down-quark Yukawa matrix. Satisfying the measured values of both θ_{13} and the electron mass restricts the number of solutions, underlying the importance of the recent measurement of the reactor angle.

^{*}E-mail: jenkile@phys.ufl.edu

[†]E-mail: mjperez@phys.ufl.edu

[‡]E-mail: ramond@phys.ufl.edu

[§]E-mail: juezhang@phys.ufl.edu

1 Introduction

The measurement of the third neutrino mixing angle θ_{13} [1, 2, 3] affords us an interesting opportunity to stringently compare the predictions of flavor models to observation, and opens the door for substantial CP violation in the neutrino sector.¹ In this work, we consider the ramifications of this result, along with those of other flavor measurements in the quark, charged-lepton, and neutrino sectors within the context of Grand Unified Theories (GUTs).

GUT patterns and relationships between quark and lepton Yukawa couplings, together with constraints from flavor observables, can be quite a powerful combination: up- and down-quark Yukawa matrices are related by the Cabibbo-Kobayashi-Maskawa (CKM) matrix; GUTs then suggest close relations between the down-quark and charged-lepton mass matrices; the Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix then relates the charged-lepton Yukawa matrix to the neutrino mass matrix, itself a function of the high-scale Majorana mass matrix and the neutral Yukawa matrix; finally the latter can be related back to the up-quark Yukawa coupling via GUT-inspired relations. We call this set of relations the “Flavor Ring”.

In a previous work [5], we investigated the relation between the up-quark and seesaw sectors of the Flavor Ring within the framework of a $(\mathcal{Z}_7 \times \mathcal{Z}_3)$ flavor symmetry [6]. In this paper, we close the ring by presenting the results of a numerical study of the interplay between the CKM parameters, charged lepton masses, and neutrino mixing angles. Given a down-quark Yukawa matrix, possible charged-lepton Yukawa matrices can be deduced through $SU(5)$ GUT relations and the use of Georgi-Jarlskog (GJ) insertions [7].

In the numerical search, we first assume a diagonal up-quark Yukawa matrix, motivated by the $(\mathcal{Z}_7 \times \mathcal{Z}_3)$ flavor symmetry. The down-quark Yukawa matrix is then determined by the CKM matrix, GUT-scale masses, and a right-handed unitary matrix, whose form we vary. Through $SU(5)$ relations we then obtain the charged-lepton Yukawa matrix, including up to eight GJ factors. We diagonalize this matrix to obtain lepton masses and its left-handed unitary matrix. To make contact with the MNSP matrix, we assume the seesaw neutrino mixing matrix (neglecting phases) to be of the Tri-bimaximal (TBM) or Bimaximal (BM) forms which share the interesting feature of $(\mu - \tau)$ symmetry and vanishing seesaw one-three mixing, implying that CP violation in the lepton sector must originate in the quark sector.

The search inputs are the rotation angles in the right-handed down-quark unitary matrix, and all possible ways of assigning GJ factors; outputs are the MNSP angles, lepton mass ratios, and the MNSP Jarlskog invariant. We retain as solutions those outputs which are compatible with experimental constraints.

Our results display some interesting features: first, there are no solutions for which the down-quark Yukawa matrix is symmetric; secondly, satisfying both the electron-to-muon mass ratio m_e/m_μ and $\theta_{13} \sim 9^\circ$ severely constrains the down-quark and charged lepton Yukawa matrices, and is key to their asymmetry.

¹See [4] for reviews on neutrinos and flavor model building.

If θ_{13} had been measured to be $\sim 3^\circ$ as predicted in many models, either the original Georgi-Jarlskog scenario or a symmetric down-quark Yukawa would have sufficed to reproduce all observed data. This highlights the importance of the measurements of θ_{13} .

The rest of this paper is organized as follows. In Section 2, we describe the Flavor Ring in detail and give the relevant relations between the down-quark and charged-lepton Yukawa matrices and the neutrino mixing angles, as well as the basics of GJ insertions. The main numerical search results can be found in Section 3, where we list the search assumptions, explain the search procedure and discuss its results by presenting some instructive examples. We also give an alternative way of performing the search in Section 4. Section 5 gives our conclusions. Detailed numerical results and their interpretations can be found in the Appendices.

2 The Flavor Ring

Flavor models aim to provide a satisfactory description of the masses and mixing between the three families of the Standard Model. For the up- and down-type quarks and charged leptons, these masses and mixings are generated by the Dirac Yukawa matrices $Y^{(2/3)}$, $Y^{(-1/3)}$, and $Y^{(-1)}$. Although the origin of neutrino masses is still an open question, the seesaw mechanism [8, 9], which provides a satisfactory explanation for small neutrino masses, adds to this list two more matrices: the neutral Dirac Yukawa matrix $Y^{(0)}$ and \mathcal{M} , the Majorana mass matrix of the right-handed neutrinos.

Our aim is to provide a logical framework, using “top-down” ideas from the seesaw mechanism and GUTs, by which these matrices may be related to one another, forming a “Flavor Ring” shown in Fig. 1.

2.1 Previous Work: the Seesaw Sector

In a previous publication [5] we addressed the seesaw sector of the Flavor Ring. We assumed that the form of the Majorana matrix \mathcal{M} was determined by the family symmetry $(\mathcal{Z}_7 \rtimes \mathcal{Z}_3)$, the smallest non-Abelian discrete subgroup of $SU(3)$; it contains 21 elements and only $\mathbf{3}$, $\bar{\mathbf{3}}$, $\mathbf{1}'$, $\bar{\mathbf{1}}'$ and $\mathbf{1}$ representations.

Using $SO(10)$ as a guide, where all $SU(5)$ matter fields, $\mathbf{10}(\chi)$, $\bar{\mathbf{5}}(\psi)$ and $\mathbf{1}(N)$ are contained in the sixteen-dimensional spinor representation, the three chiral families transform as one $(\mathcal{Z}_7 \rtimes \mathcal{Z}_3)$ triplet.

The product of two triplet representations is given by,

$$\mathbf{3} \otimes \mathbf{3} = (\mathbf{3} + \bar{\mathbf{3}})_+ + \bar{\mathbf{3}}_-,$$

where the symmetric $\mathbf{3}$ is diagonal, and the $\bar{\mathbf{3}}_\pm$ are off-diagonal symmetric and antisymmetric, respectively.

Our choice was to take the Higgs fields as anti-triplets, thereby assuring a diagonal $Y^{(2/3)}$.

In $SO(10)$, it is natural to relate the Dirac Yukawa matrices,

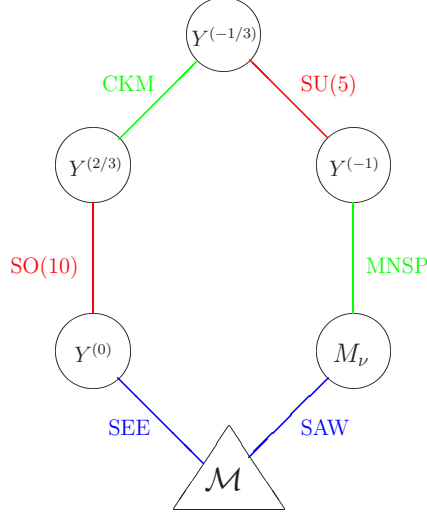


Figure 1: The “Flavor Ring”.

$$Y^{(2/3)} \sim Y^{(0)}, \quad (1)$$

where $Y^{(0)}$ appears in the numerator of the seesaw mass formula,

$$M_\nu = Y^{(0)} \mathcal{M}^{-1} Y^{(0)T} = \mathcal{U}_{\text{seesaw}} D_\nu \mathcal{U}_{\text{seesaw}}^T, \quad (2)$$

where $D_\nu = \text{diag}(m_1, m_2, m_3)$ is the diagonal light neutrino mass matrix, and $\mathcal{U}_{\text{seesaw}}$ is the seesaw neutrino mixing matrix. Eq.(1) may then seem puzzling, since the large hierarchy of the up-quark sector is not replicated amongst the light neutrinos, implying that the Majorana matrix undoes the up-quark hierarchy.

Using one linear combination of two $(\mathbf{Z}_7 \times \mathbf{Z}_3)$ -invariant dimension-five couplings (or a single dimension-six coupling), we constructed a predictive and compelling Majorana matrix with the required large correlated hierarchy; it predicts normal hierarchy and the values of neutrino masses, leads to Tri-bimaximal and Golden Ratio seesaw mixing matrices, and its correlated hierarchy is entirely generated by the vacuum values of familon fields.

The starting assumptions of the numerical search presented in this paper are then a diagonal $Y^{(2/3)}$, and either a Tri-bimaximal or Bimaximal² $\mathcal{U}_{\text{seesaw}}$. Although inspired by our previous work, the conclusions of this paper do not depend on any family symmetry.

²The results for Golden Ratio mixing will be similar to TBM aside from the value of θ_{12} . We consider Bimaximal mixing as well, since this pattern is distinct from TBM or Golden Ratio mixing.

2.2 Closing the Flavor Ring

There remains to determine the missing links required to close the flavor ring.

One is the observable leptonic mixing matrix $\mathcal{U}_{\text{MNSP}}$, which relates M_ν , (and thus $Y^{(0)}$ and \mathcal{M}), and the charged-lepton matrix $Y^{(-1)}$ through $\mathcal{U}_{\text{seesaw}}$,

$$\mathcal{U}_{\text{MNSP}} = \mathcal{U}_{-1}^\dagger \mathcal{U}_{\text{seesaw}}, \quad (3)$$

where \mathcal{U}_{-1} is determined by the charged lepton Yukawa matrix $Y^{(-1)}$.

The second is the observed CKM mixing of the quarks which connects $Y^{(2/3)}$ to $Y^{(-1/3)}$, and since $Y^{(2/3)}$ is diagonal,

$$\mathcal{U}_{\text{CKM}} = \mathcal{U}_{-1/3}, \quad (4)$$

where $\mathcal{U}_{-1/3}$ is the left-handed unitary matrix which diagonalizes $Y^{(-1/3)}$. This is enough to determine a symmetric $Y^{(-1/3)}$, which, as we shall see, does not reproduce the Daya Bay/RENO measurements.

The last step is to relate the down-quark and charged-lepton sectors. Such a relationship is natural in minimal $SU(5)$, where the Yukawa coupling $\psi\chi H_d$ to a $\mathbf{\bar{5}}$ Higgs predicts,

$$Y^{(-1)} = Y^{(-1/3)T}, \quad (5)$$

providing the final link. Although it successfully predicts $m_b = m_\tau$ at unification, it also yields $m_e = m_d$ and $m_\mu = m_s$, while renormalization group-running favors

$$m_e = m_d/3, \quad m_\mu = 3m_s. \quad (6)$$

An elegant solution to this problem, the Georgi-Jarlskog mechanism [7], is to add one additional $\mathbf{45}$ Higgs coupling, $\psi\chi H_{\mathbf{45}}$. Its original realization was through Yukawa couplings of the form,

$$Y_{ij}^{(\mathbf{\bar{5}})} \psi^i \chi^j H_d, \quad Y_{ij}^{(\mathbf{\bar{5}})} = \begin{pmatrix} 0 & a' & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \quad Y_{ij}^{(\mathbf{45})} \psi^i \chi^j H_{\mathbf{45}}, \quad Y_{ij}^{(\mathbf{45})} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Clebsch-Gordan coefficients of the $\mathbf{45}$ coupling give an extra factor of -3 in the (22) position of $Y^{(-1)}$. With the assumption that $a \sim a' \ll b \ll c$, this reproduces Eq.(6).

Although it predicts the correct light charged-lepton mass ratio, this simple model makes another prediction, namely that the mixing between the light charged leptons is given by $\theta_{12}^e = \lambda/3$, where λ is the Cabibbo angle; with TBM or BM seesaw mixing, this corresponds to

$$\theta_{13} = \frac{\lambda}{3\sqrt{2}} \approx 3^\circ,$$

smaller than the current data.

However, **45** couplings can in principle occur in any matrix element; in this work we explore all such possible couplings. As an example, if the **45** coupling dominates the (12) and (23) entries, we would have

$$Y^{(-1/3)} \sim \begin{pmatrix} G & \\ & J \end{pmatrix}, \text{ then } Y^{(-1)} \sim \begin{pmatrix} -3G & \\ & -3J \end{pmatrix}, \quad (7)$$

In the following, we use this mechanism to relate $Y^{(-1/3)}$ to $Y^{(-1)}$ and close the flavor ring.

3 Numerical Search for Allowed $Y^{(-1/3)}$ and $Y^{(-1)}$

In this section, we describe our numerical search for experimentally allowed charge $(-1/3)$ and (-1) Yukawa matrices, constrained by the following set of assumptions:

- The up-quark Yukawa matrix $Y^{(2/3)}$ is diagonal. Thus the CKM mixing matrix comes from the diagonalization of $Y^{(-1/3)}$,

$$Y^{(-1/3)} = \mathcal{U}_{\text{CKM}} \mathcal{D}_d \mathcal{V}^\dagger, \quad (8)$$

where \mathcal{D}_d is the diagonal down-quark mass matrix, and \mathcal{V} is an hitherto arbitrary unitary right-handed matrix.

- To simplify the search, we assume that this right-handed unitary matrix contains no phases,³ and is parametrized as,

$$\mathcal{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (9)$$

where $c_{ij} = \cos \beta_{ij}$, $s_{ij} = \sin \beta_{ij}$.

- As suggested by $SU(5)$, the charged-lepton Yukawa matrix $Y^{(-1)}$, is the transpose of $Y^{(-1/3)}$, up to GJ entries.
- We assume seesaw neutrino mixing matrices, with no phases as well, of the Tribimaximal or Bimaximal forms:

$$\mathcal{U}_{\text{seesaw}} = \begin{pmatrix} \cos \theta_{12}^\nu & -\sin \theta_{12}^\nu & 0 \\ \frac{1}{\sqrt{2}} \sin \theta_{12}^\nu & \frac{1}{\sqrt{2}} \cos \theta_{12}^\nu & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \sin \theta_{12}^\nu & \frac{1}{\sqrt{2}} \cos \theta_{12}^\nu & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (10)$$

³Although we neglect phases in \mathcal{V} , we include for completeness a general phase analysis in Appendix C.

where $\theta_{12}' = \arctan(1/\sqrt{2})$ for TBM mixing, and $\theta_{12}' = 45^\circ$ for BM mixing. In this parametrization, all TBM angles lie in the fourth quadrant. This choice is arbitrary; for more details, see Appendix D.

In absence of phases in \mathcal{V} and $\mathcal{U}_{\text{seesaw}}$, phases in $Y^{(-1)}$, obtained from the CKM matrix, lead to the CP-violating phase in the MNSP matrix. However, when searching for solutions with a symmetric $Y^{(-1/3)}$, we consider all possible phases in $\mathcal{U}_{\text{seesaw}}$; see later.

3.1 Search Preliminaries

The search inputs, expressed in terms of the Wolfenstein parametrization, are:

- The CKM matrix,

$$\mathcal{U}_{\text{CKM}} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \quad (11)$$

with GUT scale values [10], $\lambda = 0.227$, $\rho = 0.22$, and $\eta = 0.33$. In supersymmetry, A is the parameter most sensitive to $\tan\beta$. Its benchmark value is $A = 0.77$, which corresponds to $\tan\beta = 10$. For reference, $A = 0.72$ for $\tan\beta = 50$. In most cases we fix $A = 0.77$, but we provide some examples of how varying A can affect the search results.

- The down-quark mass ratios, allowing for the different signs of (m_d, m_s) , labelled as (\pm, \pm) ,

$$\mathcal{D}_d = m_b \begin{pmatrix} \pm \frac{\lambda^4}{3} & & \\ & \pm \frac{\lambda^2}{3} & \\ & & 1 \end{pmatrix}, \quad (12)$$

which ensures the Gatto-Sartori-Tonin relation [11]

$$\lambda = \sqrt{\frac{m_d}{m_s}},$$

compatible with the GUT scale ranges [10],

$$0.046 \leq \left| \frac{m_d}{m_s} \right| \leq 0.058, \quad 0.008 \leq \left| \frac{m_s}{m_b} \right| \leq 0.021. \quad (13)$$

Parameter	Best fit	1σ range	3σ range
θ_{12}	33.7°	$32.6^\circ - 34.8^\circ$	$30.6^\circ - 36.8^\circ$
θ_{23} (NH)	40.7°	$39.1^\circ - 42.4^\circ$	$36.7^\circ - 53.2^\circ$
θ_{23} (IH)	41.4°	$39.7^\circ - 44.8^\circ \oplus 46.8^\circ - 51.4^\circ$	$37.0^\circ - 54.3^\circ$
θ_{13} (NH)	8.80°	$8.45^\circ - 9.21^\circ$	$7.65^\circ - 9.92^\circ$
θ_{13} (IH)	8.89°	$8.49^\circ - 9.23^\circ$	$7.67^\circ - 9.97^\circ$

Table 1: Neutrino mixing angles from global fit results [12]. NH stands for normal hierarchy, IH for inverted hierarchy.

3.2 Search Procedure

The numerical search, using Mathematica,⁴ proceeds as follows:

- The first step consists of providing an input $Y^{(-1/3)}$. We search by the number of non-zero angles β_{ij} in its right-handed rotation matrix \mathcal{V} , see Eq.(9).
- With \mathcal{V} and thus $Y^{(-1/3)}$ specified, we consider all $Y^{(-1)}$'s obtained from $Y^{(-1/3)}$ by transposition and given assignments of GJ factors. These $Y^{(-1)}$'s are organized in terms of the number of GJ factors they contain.
- The outputs of the search are the charged lepton mass ratios, the neutrino mixing angles, and the MNSP Jarlskog invariant, obtained from the diagonalization of $Y^{(-1)}$.
- A “successful solution” is defined as one that yields both mass ratios within their allowed ranges at the GUT scale [10],

$$0.0046 \leq |m_e/m_\mu| \leq 0.0050, \quad 0.048 \leq |m_\mu/m_\tau| \leq 0.06, \quad (14)$$

and leptonic mixing angles which fall within the “best fit” ranges [12], shown in Table 1.⁵

This table shows that at 1σ , some but not all solutions may discriminate between normal and inverted neutrino hierarchies; however, at 3σ there is substantial overlap.

⁴The interested reader may find the corresponding notebook and their documentation as ancillary files on the arXiv.

⁵Our search is for acceptable GUT scale parameters. As explored in [13], RG effects can be appreciable depending on the neutrino mass spectrum. In our previously considered model, with TBM mixing, the lightest neutrino mass is sufficiently small so that these effects are negligible; for a more general neutrino spectrum, additional care should be given.

Solutions may be characterized by how close their numerical outputs are to these best fit values. At both extremes, an “excellent” solution is one for which all mixing angles fall within their 1σ range, a “poor” solution one for which the mixing angles only agree at 3σ . In between, subtleties arise, as there may be solutions for which two of the mixing angles are within 1σ but the third agrees only at 3σ .

The next stage in the search is guided by the reasonable expectation that as we increase the number of angles in \mathcal{V} , a better fit to the experimental values may be found. Accordingly, when one or two of the three input parameters β_{ij} are zero, we consider a “solution” to be one for which the neutrino mixing angles fall within the 3σ range. If all three input angles are non-zero, we instead demand that the neutrino mixing angles fall within the 1σ range.

- A technical remark.

We first perform coarse scans over the input angles β_{ij} , with a step size of 1° to search for solutions. The results fall into three categories.

The first category are solutions, for which all output parameters fall within the appropriate ranges described above.

The next are “close solutions” for which the output mass ratios and mixing angles agree within some tolerance beyond their bounds. For these “close solutions”, a finer scan is performed by varying β_{ij} with a step size of 0.1° to see if agreement with the appropriate bounds can be achieved.

In the last category are the null results, whose outputs lie outside of the appropriate bounds even when a tolerance is included. For these, no further tuning of the β_{ij} may bring the outputs within their appropriate ranges.

3.3 First Result: $Y^{(-1/3)}$ is Not Symmetric

We begin by searching for symmetric $Y^{(-1/3)}$, which, under our assumption of a diagonal $Y^{(2/3)}$, is totally determined,

$$Y^{(-1/3)} = \mathcal{U}_{\text{CKM}} \mathcal{D}_d \mathcal{U}_{\text{CKM}}^T, \quad (15)$$

up to different signs for m_d and m_s .

When we conduct the numerical search using this matrix as an input, we find *no* pattern of GJ insertions that reproduces the lepton masses and mixings. The same conclusion holds even if we relax two of our previous assumptions; varying the A parameter from 0.6 to 0.8, and allowing for additional phases to appear in $\mathcal{U}_{\text{seesaw}}$ for both TBM and BM mixings produce no additional solutions; see Appendix C.

As a curiosity, we investigated further the reason why there are no solutions; for those $Y^{(-1)}$ with a satisfactory value of m_e/m_μ , θ_{13} is around 3° . This illustrates the constraining and important role of the recent measurement of θ_{13} , and its tension with m_e/m_μ .

3.4 One-Angle Solutions

We therefore look for *asymmetric* $Y^{(-1/3)}$, beginning with the simplest case, where only one of the rotation angles in \mathcal{V} is non-zero.

- There are no one-angle solutions for Bimaximal mixing.
- There are one-angle solutions for Tri-bimaximal mixing only for $\beta_{13} \neq 0$, and with two and three GJ insertions.

These solutions are narrowly centered around $\beta_{13} = 183^\circ$ for $(++)$, and $\beta_{13} = 3^\circ$ for $(-+)$. Solutions for $(--)$ and $(+-)$ are the same as $(-+)$ and $(++)$ respectively, with the value of β_{13} shifted by 180° .

Below we give two examples of such solutions.

Example I: (seesaw) TBM mixing, $(+, +)$, $\beta_{13} = 183^\circ$, and two GJ factors in the (21) and (22) elements of $Y^{(-1/3)}$ yield

$$\begin{aligned} \theta_{12} &= 31.2^\circ, & \theta_{23} &= 44.6^\circ, & \theta_{13} &= 8.29^\circ, \\ \frac{m_e}{m_\mu} &= 0.0051, & \frac{m_\mu}{m_\tau} &= 0.051, & J_{\text{MNSP}} &= 3.6 \times 10^{-7}, \end{aligned} \quad (16)$$

where J_{MNSP} is the predicted Jarlskog invariant in the MNSP matrix. There exists a similar solution with the same β_{13} and an additional GJ factor in the small (32) element of $Y^{(-1/3)}$.

They are “close solutions”, with m_e/m_μ slightly above its upper bound. Their predicted value of θ_{12} is lower than the best fit, falling slightly outside of the 2σ range, but agreeing at 3σ . The other two mixing angles also fall outside the 1σ range, with a small caveat; if the neutrino hierarchy is inverted, θ_{23} agrees at 1σ . These are examples of solutions that discriminate between the two hierarchies, see Table 1.

A finer scan of β_{13} around 183° with a step size of 0.01° reveals that, for a 1° variation in β_{13} , the ratio m_e/m_μ varies by at most ± 0.003 . For $183.03^\circ < \beta_{13} < 183.12^\circ$, ($A = 0.77$), we find

$$\begin{aligned} 31.0^\circ < \theta_{12} < 31.1^\circ, & \quad \theta_{23} = 44.6^\circ, & \quad 8.37^\circ < \theta_{13} < 8.63^\circ, \\ 0.0046 < \frac{m_e}{m_\mu} < 0.0050, & \quad \frac{m_\mu}{m_\tau} = 0.051. \end{aligned}$$

A second possibility is to fix β_{13} and instead vary A . For example, setting $\beta_{13} = 183^\circ$, and $0.79 < A < 0.81$, we find,

$$\begin{aligned} 30.9^\circ < \theta_{12} < 31.1^\circ, & \quad \theta_{23} = 44.6^\circ, & \quad 8.32^\circ < \theta_{13} < 8.48^\circ, \\ 0.0047 < \frac{m_e}{m_\mu} < 0.0050, & \quad \frac{m_\mu}{m_\tau} = 0.051, \end{aligned}$$

and the higher values for A imply $\tan \beta < 10$.

Analysis

The existence of a one-angle solution is of special interest, since all mass ratios and mixing angles are fit by one extra input parameter, β_{13} . In addition, its numerical value suggests a parametrization in terms of λ , with the angle in radians,

$$\beta_{13} = \pi + B\lambda^2,$$

where $B \approx 1$. \mathcal{V} is then given by

$$\mathcal{V} = \begin{pmatrix} -1 + \frac{B^2\lambda^4}{2} & 0 & -B\lambda^2 \\ 0 & 1 & 0 \\ B\lambda^2 & 0 & -1 + \frac{B^2\lambda^4}{2} \end{pmatrix} + \mathcal{O}(\lambda^6). \quad (17)$$

Accordingly, $Y^{(-1/3)}$ takes the form

$$Y^{(-1/3)} = \begin{pmatrix} -\frac{1}{3}\lambda^4 - AB\lambda^5(-i\eta + \rho) & \frac{\lambda^3}{3} & -A\lambda^3(-i\eta + \rho) \\ -AB\lambda^4 + \frac{\lambda^5}{3} & \frac{1}{3}\lambda^2 \left(1 - \frac{\lambda^2}{2}\right) & -A\lambda^2 \\ -B\lambda^2 & -\frac{A\lambda^4}{3} & -1 + \frac{B^2\lambda^4}{2} \end{pmatrix} + \mathcal{O}(\lambda^6),$$

and $Y^{(-1)}$ is given by

$$Y^{(-1)} = \begin{pmatrix} -\frac{1}{3}\lambda^4 - AB\lambda^5(-i\eta + \rho) & 3AB\lambda^4 - \lambda^5 & -B\lambda^2 \\ \frac{\lambda^3}{3} & -\lambda^2 \left(1 - \frac{\lambda^2}{2}\right) & -\frac{A\lambda^4}{3} \\ -A\lambda^3(-i\eta + \rho) & -A\lambda^2 & -1 + \frac{B^2\lambda^4}{2} \end{pmatrix} + \mathcal{O}(\lambda^6). \quad (18)$$

Its diagonalization yields the charged-lepton mass ratios and the mixing angles θ_{ij}^e in the left-handed unitary matrix \mathcal{U}_{-1} .⁶

$$\begin{aligned} m_\mu/m_\tau &= \lambda^2 + \mathcal{O}(\lambda^4), \\ m_e/m_\mu &= \frac{1}{3}\lambda^2 - \frac{4}{3}AB\lambda^3 + \mathcal{O}(\lambda^4), \\ \theta_{12}^e &= 4AB\lambda^2 + \mathcal{O}(\lambda^3), \\ \theta_{13}^e &= -B\lambda^2 + \mathcal{O}(\lambda^4), \\ \theta_{23}^e &= \frac{A\lambda^4}{3} + \mathcal{O}(\lambda^8). \end{aligned} \quad (19)$$

⁶Some caution must be taken when obtaining θ_{12}^e . It is not simply the ratio of the (12) and (22) entries of $Y^{(-1)}$; an “ $AB\lambda^4$ ” term coming from the mixing of θ_{13}^e must be taken out of the (12) entry before evaluating the ratio.

The GJ factor in the (22) entry results in the correct value for m_μ/m_τ , while for m_e/m_μ the input angle β_{13} also plays an important role. Since the Tribimaximal seesaw mixing matrix is a good approximation to the data, we expect the three mixing angles θ_{ij}^e in \mathcal{U}_{-1} to be small. Note that θ_{12}^e now differs from $\beta_{12} = 0$ because of the placements of GJ factors.

Using Eq.(3), we obtain the MNSP mixing angles,

$$\begin{aligned}\theta_{23} &= \frac{\pi}{4} + \mathcal{O}(\lambda^4), \\ \theta_{13} &= \frac{(4A+1)B\lambda^2}{\sqrt{2}} + \mathcal{O}(\lambda^3), \\ \theta_{12} &= \arctan \frac{1}{\sqrt{2}} - \delta\theta_{12},\end{aligned}\tag{20}$$

where

$$\delta\theta_{12} = \frac{(4A-1)B\lambda^2}{\sqrt{2}} + \mathcal{O}(\lambda^3).\tag{21}$$

These analytical expressions for the mixing angles agree with the numerical values,

$$\theta_{23} \approx 45^\circ, \quad \theta_{13} \approx 8.6^\circ, \quad \theta_{12} \approx 30.9^\circ.$$

Since m_e/m_μ , θ_{13} and θ_{12} all depend on a single input angle β_{13} , one can derive two sum rules among them,

$$\begin{aligned}\arctan \frac{1}{\sqrt{2}} &\approx \theta_{12} + \frac{4A-1}{4A+1}\theta_{13} \approx \theta_{12} + \frac{1}{2}\theta_{13}, \\ \frac{m_e}{m_\mu} &\approx \frac{\lambda^3}{3} + \frac{\sqrt{2}\lambda}{3} \left(\frac{\lambda}{\sqrt{2}} - \theta_{13} \right).\end{aligned}\tag{22}$$

The second relation between m_e/m_μ and θ_{13} is quite interesting; since the measured value of θ_{13} is close to $\lambda/\sqrt{2}$, the order λ^2 terms in m_e/m_μ cancel, resulting in a single $\lambda^3/3$ term, which brings m_e/m_μ within the allowed range.

A naive use of $SU(5)$ and $SO(10)$ relations, assuming a symmetric $Y^{(-1/3)}$, and neglecting the Georgi-Jarlskog contributions, was found to yield $\theta_{13} = \lambda/\sqrt{2}$ [14] in the TBM case, but at the expense of a much too large electron mass. Conversely, many subsequent models found that starting from the observed electron mass, the same assumptions yielded $\theta_{13} = \lambda/3\sqrt{2}$. Only an asymmetric $Y^{(-1/3)}$ can relieve the tension between m_e/m_μ and θ_{13} .

Example II: (seesaw) TBM mixing, $(-, +)$, $\beta_{13} = 3^\circ$, and two GJ insertions in the (22) and (23) elements of $Y^{(-1/3)}$, yield:

$$\begin{aligned}\theta_{12} &= 30.5^\circ, \quad \theta_{23} = 44.7^\circ, \quad \theta_{13} = 8.88^\circ, \\ \frac{m_e}{m_\mu} &= 0.0039, \quad \frac{m_\mu}{m_\tau} = 0.050, \quad J_{\text{MNSP}} = 4.3 \times 10^{-7}.\end{aligned}\tag{23}$$

It is another “close” solution, for which agreement with the allowed range of m_e/m_μ is attained with $0.72 < A < 0.74$, this time finding:

$$30.9^\circ < \theta_{12} < 31.0^\circ, \quad \theta_{23} = 44.7^\circ, \quad 8.34^\circ < \theta_{13} < 8.50^\circ, \quad (24)$$

$$(25)$$

$$0.0046 < \frac{m_e}{m_\mu} < 0.0050, \quad \frac{m_\mu}{m_\tau} = 0.050. \quad (26)$$

This solution tends to favor lower values of A ($\tan \beta > 10$). It also predicts a low value of θ_{12} , very close to the 3σ bound, and favors the inverted hierarchy; its value for θ_{13} however, lies very close to the best fit value. As in the previous example, and for the same reason, a solution with three GJ insertions exists with the additional factor in the (32) element of $Y^{(-1/3)}$.

For this example, the numerical value of β_{13} suggests a slightly different parametrization,

$$\beta_{13} = B\lambda^2,$$

with $B \approx 1$. However, we find that a similar analysis to that given for Example I leads to exactly the same sum rules Eqs. (22).

It is quite remarkable that given our assumptions, a single free parameter is able to reasonably fit all three mixing angles and the two mass ratios.

We next turn to the cases when two of the rotation angles in \mathcal{V} are non-zero.

3.5 Two-Angle Solutions

With two input parameters, we find a larger set of solutions with a variety of GJ factors. Solutions now exist for both Tri-bimaximal and Bimaximal seesaw mixing schemes. All four sign combinations of (m_d, m_s) can be related to one another by shifting input angles, and only one case is independent. We focus on the (+,+) case.

Tri-bimaximal Seesaw Mixing

For TBM mixing, (+,+), and $A = 0.77$, we find solutions with $\beta_{23} = 0$ and $\beta_{12} = 0$ only; there are no two-angle solutions with $\beta_{13} = 0$. Compared to the one angle case, the solutions occur for a wide range of angles and for a diverse set of GJ patterns. Surprisingly, however, with two angles we are still unable to find a solution for which all mixing angles agree at the 1σ level.

A complete list of solutions with two angles and TBM mixing may be found in the tables of Appendix A.

Even for two angles we find the parameter space of solutions to be “small”, with specific regions singled out, as can be seen in Fig.2. A close inspection of the figure shows that solutions occur for β_{ij} close to an axis, and there are more solutions with $\beta_{23} = 0$, than with $\beta_{12} = 0$.

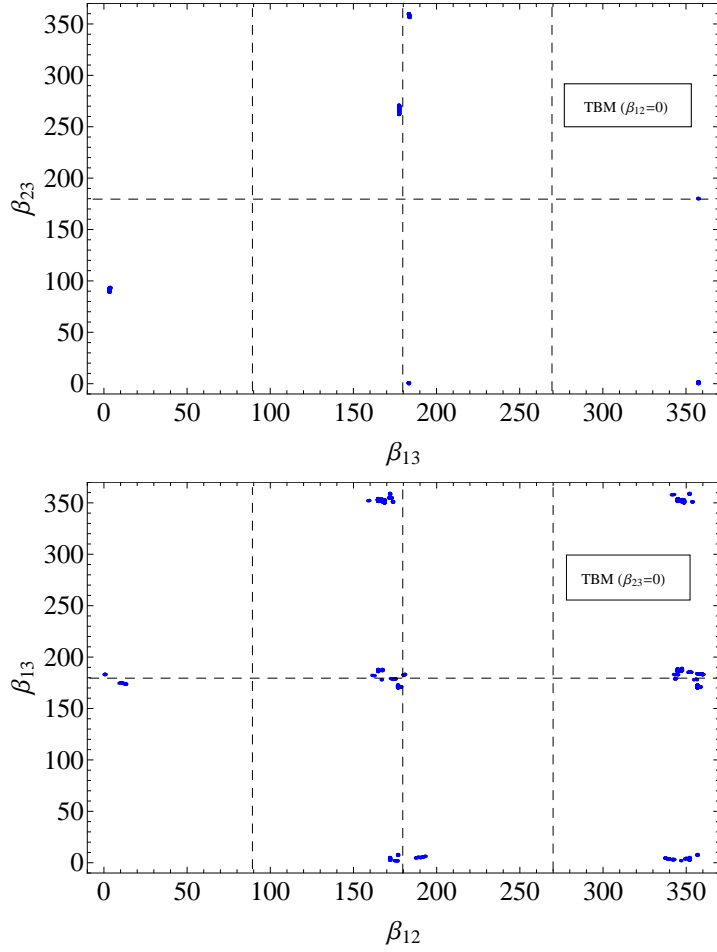


Figure 2: Allowed parameter space for TBM mixing with two non-zero input angles β_{ij} . The first figure gives the allowed values of β_{23} and β_{13} with $\beta_{12} = 0$, while the bottom figure gives the allowed values of β_{12} and β_{13} with $\beta_{23} = 0$. As noted in the text, no solutions exist with $\beta_{13} = 0$.

Since the allowed patterns of inserting GJ factors are relevant for model building, we provide a list below with respect to the number of GJ factors for both $\beta_{23} = 0$ and $\beta_{12} = 0$.

- When $\beta_{23} = 0$, we find solutions with up to six GJ factors.
 - One GJ factor:

$$\left(\begin{array}{c} \times \\ \times \end{array} \right). \quad (27)$$

– Two GJ factors:

$$\begin{pmatrix} \times & \times \\ & \end{pmatrix}, \quad \begin{pmatrix} & \times \\ & \times \end{pmatrix}, \quad \begin{pmatrix} & & \times \\ & \times & \times \end{pmatrix}. \quad (28)$$

– Three GJ factors:

$$\begin{pmatrix} \times & \times \\ & \times \end{pmatrix}, \quad \begin{pmatrix} & \times \\ \times & \times \end{pmatrix}, \quad \begin{pmatrix} \times & & \times \\ & \times & \end{pmatrix}, \\ \begin{pmatrix} & & \times \\ \times & \times & \times \end{pmatrix}, \quad \begin{pmatrix} & \times & \times \\ & \times & \times \end{pmatrix}, \quad \begin{pmatrix} \times & & \\ \times & \times & \end{pmatrix}. \quad (29)$$

– Four GJ factors:

$$\begin{pmatrix} & \times \\ \times & \times \\ & \times \end{pmatrix}, \quad \begin{pmatrix} \times & & \times \\ & \times & \\ & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \times \\ \times & \times & \\ & \times & \end{pmatrix}, \\ \begin{pmatrix} & \times & \times \\ \times & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \\ \times & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & \times \\ \times & \times \end{pmatrix}, \\ \begin{pmatrix} \times & \times & \times \\ \times & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \\ \times & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \times \\ & \times & \times \end{pmatrix}, \\ \begin{pmatrix} \times & & \\ \times & \times & \times \end{pmatrix}. \quad (30)$$

– Five GJ factors:

$$\begin{pmatrix} \times & \times \\ & \times & \times \\ \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \times \\ \times & \times & \\ & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \\ \times & \times & \times \\ \times & \end{pmatrix}, \\ \begin{pmatrix} \times & \times \\ \times & \times \\ \times & \times \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & \\ & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \times \\ \times & \times & \times \\ & \times & \end{pmatrix}, \\ \begin{pmatrix} \times & & \times \\ \times & \times & \times \end{pmatrix}, \quad \begin{pmatrix} \times & & \\ \times & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & \end{pmatrix}, \\ \begin{pmatrix} & \times & \times \\ \times & \times & \times \\ & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ & \times & \times \end{pmatrix}, \quad \begin{pmatrix} & \times & \times \\ \times & \times & \times \end{pmatrix}. \quad (31)$$

– Six GJ factors:

$$\begin{pmatrix} \times & \times & \\ & \times & \times \\ \times & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \times \\ \times & \times & \\ & \times & \end{pmatrix},$$

$$\begin{pmatrix} \times & \times & \times \\ & \times & \times \\ \times & \times & \end{pmatrix}, \quad \begin{pmatrix} \times & & \times \\ \times & \times & \times \\ & \times & \end{pmatrix}. \quad (32)$$

• When $\beta_{12} = 0$, there are solutions with only two and three GJ factors.

– Two GJ factors:

$$\begin{pmatrix} \times & & \\ & \times & \\ & & \end{pmatrix}, \quad \begin{pmatrix} & \times & \times \\ & & \\ & & \end{pmatrix}, \quad \begin{pmatrix} \times & \times & \\ & \times & \\ & & \end{pmatrix}. \quad (33)$$

– Three GJ factors:

$$\begin{pmatrix} \times & \times & \\ & \times & \\ & & \end{pmatrix}, \quad \begin{pmatrix} & \times & \times \\ & \times & \\ & & \end{pmatrix}, \quad (34)$$

$$\begin{pmatrix} \times & & \times \\ & \times & \\ & & \times \end{pmatrix}, \quad \begin{pmatrix} & \times & \times \\ & \times & \\ & & \times \end{pmatrix}. \quad (35)$$

Among these patterns, two are noteworthy. The first one is for $\beta_{23} = 0$, with one GJ factor in the (22) element of $Y^{(-1/3)}$. Surprisingly, the placement of the GJ factor coincides with the original realization of the GJ relations, but this time with a compatible small value of θ_{13} . The second one has two GJ factors in off-diagonal entries, and $\beta_{12} = 0$.

For these reasons, we provide numerical details for these two cases.

• **Example III:** (seesaw) TBM mixing, (+,+), $A = 0.77$, and

$$\beta_{12} = 168^\circ, \quad \beta_{13} = 352^\circ, \quad \beta_{23} = 0^\circ, \quad (36)$$

with one GJ factor in the (22) element of $Y^{(-1/3)}$, yields,

$$\theta_{12} = 32.6^\circ, \quad \theta_{23} = 45.1^\circ, \quad \theta_{13} = 8.68^\circ,$$

$$\frac{m_e}{m_\mu} = 0.0054, \quad \frac{m_\mu}{m_\tau} = 0.049, \quad J_{\text{MNSP}} = 9.6 \times 10^{-8}. \quad (37)$$

We note that for this example, better agreement with the best fit values of θ_{12} and θ_{13} is achieved. Only θ_{23} , the mixing angle with the largest uncertainty, is outside the 1σ range.

Analysis

Similar to the one-angle solution, we want to derive analytical formulae for the charged lepton mass ratios and neutrino mixing angles in terms of the input angles. With β_{12} and β_{13} close to the x-axis, one can parametrize $\beta_{12} = \pi - C\lambda$ and $\beta_{13} = B\lambda$, and write \mathcal{V} as

$$\mathcal{V} = \begin{pmatrix} 1 - \frac{B^2\lambda^2}{2} & 0 & B\lambda \\ 0 & 1 & 0 \\ -B\lambda & 0 & 1 - \frac{B^2\lambda^2}{2} \end{pmatrix} \begin{pmatrix} -1 + \frac{C^2\lambda^2}{2} & C\lambda & 0 \\ -C\lambda & -1 + \frac{C^2\lambda^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\lambda^3), \quad (38)$$

where numerically $C = 0.92$ and $B = -0.62$. The down-quark and charged lepton Yukawa matrices $Y^{(-1/3)}$ and $Y^{(-1)}$ follow, and diagonalizing $Y^{(-1)}$ yields the mass ratios of the charged leptons,

$$\begin{aligned} m_\mu/m_\tau &= \lambda^2 + \mathcal{O}(\lambda^4), \\ m_e/m_\mu &= \left(\frac{4}{9}C - \frac{1}{3}\right)\lambda^2 + \mathcal{O}(\lambda^4) \approx \frac{1}{9}\lambda^2, \end{aligned} \quad (39)$$

coincidentally the Georgi-Jarlskog value. The mixing angles in \mathcal{U}_{-1} can also be found, and one can derive the following neutrino mixing angles in the MNSP matrix by using Eq.(3),

$$\begin{aligned} \theta_{23} &= \frac{\pi}{4} + \mathcal{O}(\lambda^4), \\ \theta_{13} &= \frac{(C - 3B)\lambda}{3\sqrt{2}} + \mathcal{O}(\lambda^3), \\ \theta_{12} &= \arctan \frac{1}{\sqrt{2}} - \delta\theta_{12}, \end{aligned} \quad (40)$$

where

$$\delta\theta_{12} = -\frac{(C + 3B)\lambda}{3\sqrt{2}} + \mathcal{O}(\lambda^3), \quad (41)$$

corresponding to

$$\theta_{23} \approx 45^\circ, \quad \theta_{13} \approx 8.5^\circ, \quad \theta_{12} \approx 32.4^\circ. \quad (42)$$

m_e/m_μ and θ_{13} are no longer correlated, and only one sum rule can be derived among m_e/m_μ , θ_{13} and θ_{12} ,

$$\frac{m_e}{m_\mu} \approx -\frac{\lambda^2}{3} + \frac{2\sqrt{2}\lambda}{3}(\theta_{12} + \theta_{13} - \arctan \frac{1}{\sqrt{2}}). \quad (43)$$

One also notices that the order $\mathcal{O}(\lambda^2)$ terms in m_e/m_μ are not fully cancelled by taking $\theta_{13} \approx \lambda/\sqrt{2}$. Instead, an order λ^2 term, $\sim \lambda^2/9$, is left over, and causes m_e/m_μ to fit within the allowed range. This way of obtaining the correct value for m_e/m_μ is quite distinct from what we found in the one-angle solution, where the correct value for m_e/m_μ stems from an $\mathcal{O}(\lambda^3)$ term.

- **Example IV:** (seesaw) TBM mixing, $(+,+)$, $A = 0.77$, and

$$\beta_{12} = 0^\circ, \quad \beta_{13} = 3^\circ, \quad \beta_{23} = 90^\circ, \quad (44)$$

with two GJ factors in the (21) and (23) element of $Y^{(-1/3)}$, yields,

$$\begin{aligned} \theta_{12} &= 31.2^\circ, & \theta_{23} &= 44.6^\circ, & \theta_{13} &= 8.29^\circ, \\ \frac{m_e}{m_\mu} &= 0.0051, & \frac{m_\mu}{m_\tau} &= 0.051, & J_{\text{MNSP}} &= 3.6 \times 10^{-7}. \end{aligned} \quad (45)$$

This output is exactly the same as the one angle case with $\beta_{13} = 183^\circ$ and with two GJ factors in the (21) and (22) elements of $Y^{(-1/3)}$ (see Eq.(16)). This is because β_{23} , whose value is 90° , plays the role of switching the second and third column of $Y^{(-1/3)}$. It is also the reason why one can have a GJ factor in the (33) element of $Y^{(-1/3)}$ for some patterns in the case of $\beta_{12} = 0$.

Bimaximal Seesaw Mixing

For the Bimaximal case, the patterns of two-angle solutions are quite different, and come in three types with $\beta_{23} = 0$, $\beta_{12} = 0$, and $\beta_{13} = 0$, ($A = 0.77$). Likewise the patterns of GJ insertions are distinct from those of TBM mixing. We plot the parameter space for all three cases in Fig.3. A complete detailed list of these solutions and their corresponding GJ patterns can be found in the tables of Appendix A.

As for TBM mixing, we find the parameter space for two angles to be quite restricted, with input angles β_{ij} near an axis, although we note several solutions with $\sin \beta_{ij}$ significantly larger and off-axis.

From this set of solutions we give two interesting examples.

- **Example V:** (seesaw) BM mixing, $(+,+)$, $A = 0.77$, and

$$\beta_{12} = 173^\circ, \quad \beta_{13} = 165^\circ, \quad \beta_{23} = 0^\circ, \quad (46)$$

with one GJ factor in the (22) element of $Y^{(-1/3)}$, yields,

$$\begin{aligned} \theta_{12} &= 32.4^\circ, & \theta_{23} &= 46.1^\circ, & \theta_{13} &= 8.67^\circ, \\ \frac{m_e}{m_\mu} &= 0.0040, & \frac{m_\mu}{m_\tau} &= 0.050, & J_{\text{MNSP}} &= 9.9 \times 10^{-7}. \end{aligned} \quad (47)$$

The analysis of this example follows closely that of Example III, with similar conclusions about m_e/m_μ and θ_{13} .

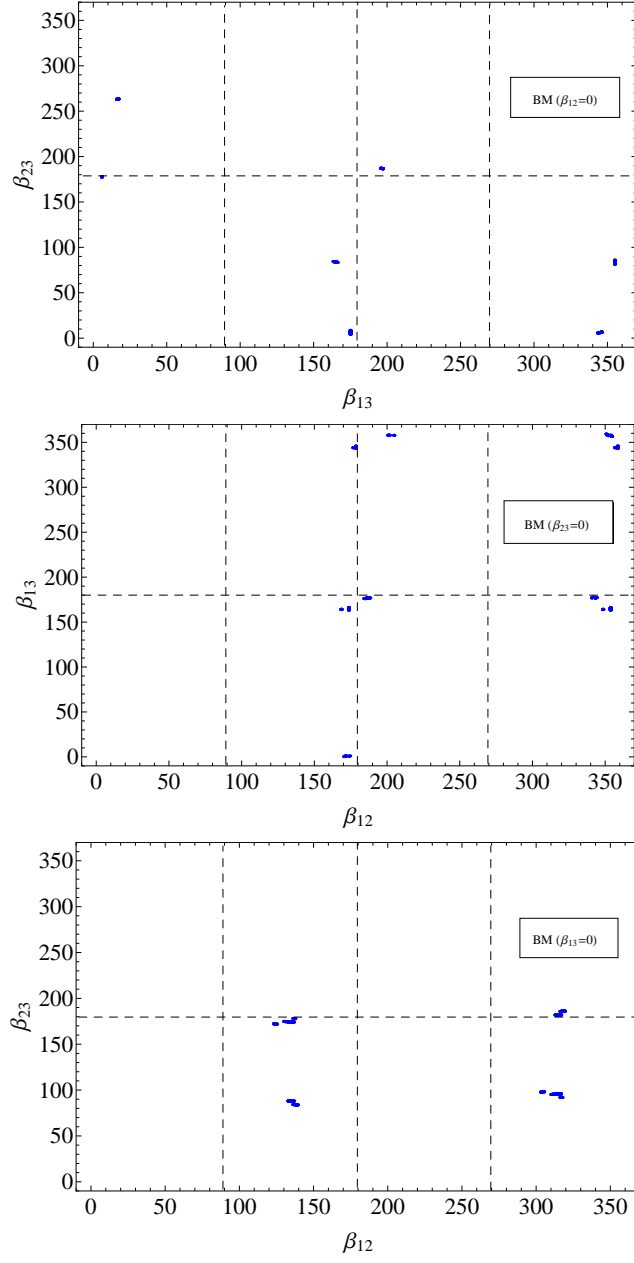


Figure 3: Allowed parameter space for BM mixing with two non-zero input angles β_{ij} .

- **Example VI:** (seesaw) BM mixing, $(+, +)$, $A = 0.77$, and

$$\beta_{12} = 314.3^\circ, \quad \beta_{13} = 0^\circ, \quad \beta_{23} = 181.8^\circ, \quad (48)$$

where β_{12} is not near an axis, with four GJ factors in the (12), (22), (23) and (32) matrix elements of $Y^{(-1/3)}$. It yields,

$$\begin{aligned} \theta_{12} &= 34.3^\circ, \quad \theta_{23} = 39.2^\circ, \quad \theta_{13} = 8.70^\circ, \\ \frac{m_e}{m_\mu} &= 0.0049, \quad \frac{m_\mu}{m_\tau} = 0.051, \quad J_{\text{MNSP}} = 2.6 \times 10^{-6}. \end{aligned} \quad (49)$$

This solution is noteworthy because all mixing angles now fall within the 1σ range, remarkably close to their best fit values. We may contrast it with TBM mixing, for which no such solutions exist.

Analysis

To investigate this solution analytically, we first parametrize the two input angles in terms of λ as,

$$\beta_{12} = -\frac{\pi}{4} - C\lambda^3, \quad \beta_{23} = \pi + D\lambda^2, \quad (50)$$

trading them for two order-one parameters, $C = 1.04$ and $D = 0.61$. As before,

$$\frac{m_\mu}{m_\tau} = \lambda^2 + \mathcal{O}(\lambda^4).$$

Similarly,

$$m_e/m_\mu = -\frac{\lambda^2}{3} + 2\sqrt{2}AD\lambda^3 - 2\sqrt{2}AD\lambda^4 + \mathcal{O}(\lambda^5). \quad (51)$$

The $\mathcal{O}(\lambda^4)$ term has to be kept because, with $D = 0.61$ and $A = 0.77$ the first two terms in this expansion nearly cancel, in an apparent numerical coincidence. We also have

$$\theta_{23} = \frac{\pi}{4} - 3D\lambda^2 + \mathcal{O}(\lambda^4), \quad (52)$$

but it does not appear easy to derive analytical expressions for the other two MNSP mixing angles θ_{12} and θ_{13} , although by numerical methods, we find

$$\theta_{13} = \frac{1}{6} + \mathcal{O}(\lambda^2), \quad \theta_{12} = \arcsin \frac{5}{6\sqrt{2}} + \mathcal{O}(\lambda^2). \quad (53)$$

Finally, we come to the most computationally challenging case, where all three angles in \mathcal{V} are non-zero.

3.6 Three-Angle Solutions

As we increase the number of input angles to three, the number of solutions increases, as does the time required to scan the full parameter space. Luckily, we may use analytical arguments to guide our search, and restrict the scan to specific regions of the β_{ij} within their full $(0, 2\pi)$ range.

That such an analysis is possible follows from the fact that the TBM and BM mixing angles are reasonably close to the best fit values of the MNSP mixing angles. Aside from subtleties involving the quadrants of the angles (see Appendix D), this allows us to conclude that the mixing angles θ_{ij}^e in \mathcal{U}_{-1} need to be close to one of the axes.

Knowing the required size of the corrections, one may expand Eq.(3) in terms of the small mixing angles of \mathcal{U}_{-1} , obtaining bounds on their magnitudes.

Translating them to bounds on the mixing angles in \mathcal{V} requires more care, as going from $Y^{(-1/3)}$ to $Y^{(-1)}$ involves both a transposition and modification by GJ factors. The complete details on how these bounds are obtained may be found in Appendix C.

To summarize our results, we find that to obtain suitable corrections, β_{23} must be at most 10° away from the axes for both TBM and BM mixing, while β_{13} can be 10° (20°) for TBM (BM); β_{12} however is unrestricted, and can assume its full $(0, 2\pi)$ range.

Following the two-angle case, we present our results in table form at the end of Appendix A. With the addition of the third input angle, we find that while the number of 1σ solutions increases, they are surprisingly still quite rare.

To get a feel for how the parameter space looks for three angles, we plot the allowed regions of β_{12} and β_{13} for both TBM and BM mixing in Figs.4-5; we distinguish the two cases by whether β_{23} is close to the x- or y-axes.

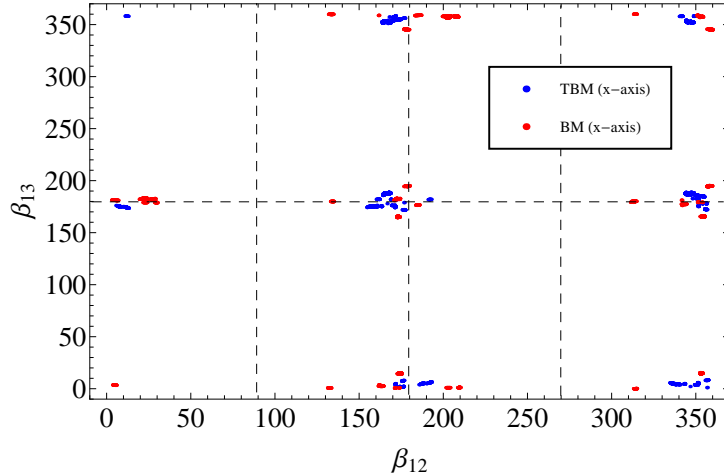


Figure 4: A plot of the allowed values of β_{13} and β_{12} for β_{23} close to the x-axis for both TBM and BM mixing.

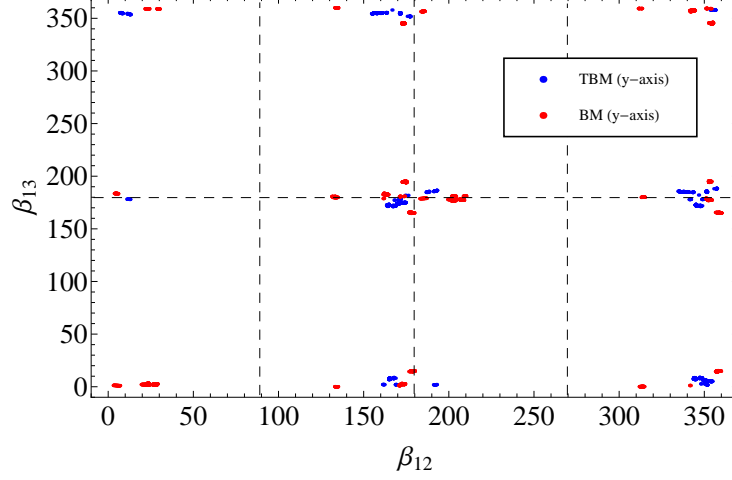


Figure 5: A plot of the allowed values of β_{13} and β_{12} for β_{23} close to the y-axis for both TBM and BM mixing.

As in the two-angle case, the number of solutions is quite small, tending to cluster around specific regions close to one of the axes. We also find that the BM solutions tend to cover a wider spread of input angles, similar to the two-angle solutions.

To close, we give an example of a solution with three angles that produces outputs in good agreements with the best fit values.

- **Example VII:** (seesaw) TBM mixing, $(+, +)$, $A = 0.77$, and

$$\beta_{12} = 169.7^\circ, \quad \beta_{13} = 354.0^\circ, \quad \beta_{23} = 176.5^\circ, \quad (54)$$

with two GJ factor in the (21) and (22) elements of $Y^{(-1/3)}$, yields,

$$\begin{aligned} \theta_{12} &= 34.4^\circ, \quad \theta_{23} = 41.3^\circ, \quad \theta_{13} = 8.73^\circ, \\ \frac{m_e}{m_\mu} &= 0.0050, \quad \frac{m_\mu}{m_\tau} = 0.059, \quad J_{\text{MNSP}} = 1.1 \times 10^{-8}. \end{aligned} \quad (55)$$

4 Almost Symmetric $Y^{(-1/3)}$

In the previous sections, we chose to introduce the asymmetry in $Y^{(-1/3)}$ by expanding \mathcal{V} away from unity. One could have equally well introduced the asymmetry by starting from a symmetric $Y^{(-1/3)}$ by taking ,

$$\mathcal{V} = \mathcal{U}_{\text{CKM}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \hat{c}_{23} & \hat{s}_{23} \\ 0 & -\hat{s}_{23} & \hat{c}_{23} \end{pmatrix} \begin{pmatrix} \hat{c}_{13} & 0 & \hat{s}_{13} \\ 0 & 1 & 0 \\ -\hat{s}_{13} & 0 & \hat{c}_{13} \end{pmatrix} \begin{pmatrix} \hat{c}_{12} & \hat{s}_{12} & 0 \\ -\hat{s}_{12} & \hat{c}_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (56)$$

where $\hat{s}_{ij} = \sin \gamma_{ij}$, and $\hat{c}_{ij} = \cos \gamma_{ij}$. The angles γ_{ij} indicate directly the deviation from symmetry. In this case, the search for one-angle solutions is akin to the generic three-angle solutions where the angles and phases obey several relations among themselves. This form may be more convenient for model building purposes.

Results are tabulated in Appendix B.

5 Summary and Conclusions

The Flavor Ring demonstrates how the simultaneous consideration of constraints on flavor observables and ideas inspired by grand unification can significantly restrict the flavor sector of the SM. While these constraints do not uniquely determine the flavor structure of the SM fermions, they reduce the parameter space sufficiently to allow one to begin to do numerical searches for solutions compatible with all the fermion GUT-scale mass ratios and quark and lepton mixing angles. The solutions found by these searches can then potentially be compared against the predictions of specific flavor models.

In this work, we have performed a search for such solutions. Inspired by a $\mathbf{Z}_7 \rtimes \mathbf{Z}_3$ flavor symmetry, we take a diagonal $Y^{(2/3)}$, which then allows us to write $Y^{(-1/3)}$ in terms of the CKM matrix, a diagonal matrix, and a single right-handed unitary matrix, \mathcal{V} . We first consider a symmetric $Y^{(-1/3)}$, setting $\mathcal{V}^\dagger = \mathcal{U}_{\text{CKM}}^T$, and then expand the search to include forms of \mathcal{V} containing one, two, and three nonzero mixing angles. For each of these cases, we obtain the corresponding $Y^{(-1)}$, including up to eight Georgi-Jarlskog insertions. We then retain those forms of $Y^{(-1)}$ which produce phenomenologically acceptable values for the neutrino mixing angles and the charged-lepton mass ratios, assuming either a BM or TBM form of the neutrino seesaw mass matrix.

We find several interesting features of the search results. A first point is that we find no symmetric forms of $Y^{(-1/3)}$ compatible with the measured value of $\theta_{13} \sim 9^\circ$, even if we include phases in $\mathcal{U}_{\text{seesaw}}$. However, many solutions exist for $\theta_{13} \sim 3^\circ$. This highlights two important points. First, our results would place significant constraints on flavor models which predict a symmetric $Y^{(-1/3)}$.⁷ Second, this underscores the importance of the recent measurement of θ_{13} in constraining models of flavor.

Next, we examine solutions with asymmetric $Y^{(-1/3)}$. Our solutions are discussed in Secs. 3 and 4, listed in Appendices A and B, and shown in Figs. 2-5. We find that the values of the angles β_{ij} cluster around the values $0, 180^\circ$, and $\pm 90^\circ$. This general feature is not surprising, as the form of the TBM and BM seesaw mixing matrices is chosen to very approximately coincide with the MNSP matrix; details can be found in Appendices C and D. Additionally, we find that the values of m_e/m_μ and θ_{13} to be particularly effective at constraining \mathcal{V} and the placement of GJ entries.

Also, we should emphasize that the general framework of combining flavor observables and ideas from grand unification can be applied somewhat more

⁷For specific models, however, this will depend upon our assumption of a diagonal $Y^{(2/3)}$.

widely than was done here. For example, the diagonal form of $Y^{(2/3)}$, chosen here due to its relevance to $\mathbf{Z}_7 \rtimes \mathbf{Z}_3$ flavor models, could be replaced with other forms on a model-specific basis. The neutrino seesaw matrix could similarly take forms other than the BM and TBM models studied here. Thus, similar, dedicated searches could be performed to constrain or possibly rule out particular models.

In summary, the consideration of flavor ideas under the umbrella of grand unification allows one to relate flavor observables from the neutrino, charged-lepton, and quark sectors in ways that can be compared to currently existing data. We have performed such a comparison under the assumption of specific up-quark Yukawa and neutrino seesaw matrices, and other, similar comparisons are ripe for investigation.

6 Acknowledgements

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A Search Results: Asymmetric $Y^{(-1/3)}$

Solutions with two angles in \mathcal{V}

All search results are presented in the form of tables. Tables 2 and 3 give the results for the TBM case, while those for the BM case can be found in tables 4, 5 and 6.

The first three entries of each table list the angles β_{ij} in \mathcal{V} for which there are solutions. The second column, labelled “GJ Entries”, indicates the location of the GJ couplings in the down-quark Yukawa matrix, and the magnitudes of entries in powers of λ .

In Tables 2 and 4 we only list the magnitudes in λ for entries in the first column of $Y^{(-1/3)}$ (transposed to save space). The entries of the last two columns do not vary much, because the angles that yield solutions are at most 20 degrees away from the x-axis. In terms of the Cabibbo angle, the range of the absolute values of the $Y^{(-1/3)}$ matrix elements are given by,

$$|Y^{(-1/3)}| \sim \begin{pmatrix} \lambda^4 - \lambda^6 & (1.0 - 1.5)\lambda^4 & (1.0 - 1.56)\lambda^4 \\ \lambda^3 - \lambda^6 & (0.31 - 0.32)\lambda^2 & (0.70 - 0.75)\lambda^2 \\ \lambda^1 - \lambda^4 & \lambda^5 & 1 \end{pmatrix}. \quad (\text{A.1})$$

For example, the first solution in Table 2 has the magnitude of the entries in $Y^{(-1/3)}$ given by

$$|Y^{(-1/3)}| = \begin{pmatrix} \lambda^{5.0} & \lambda^{3.7} & \lambda^{3.8} \\ \lambda^{4.0} & \lambda^{2.8} & \lambda^{2.2} \\ \lambda^{1.3} & \lambda^{4.9} & 1 \end{pmatrix}. \quad (\text{A.2})$$

In Tables 3, 5 and 6 we instead list the magnitudes in λ for all entries in $Y^{(-1/3)}$, as a similar pattern to that above does not occur.

The numerical outputs are given in the remaining columns of the tables: the mass ratio m_e/m_μ and the mixing angles in \mathcal{U}_{-1} , including some phases, which will be explained in Appendix C. The MNSP mixing angles and the predicted Jarlskog invariant in $\mathcal{U}_{\text{MNSP}}$ appear in the last four columns.

All the angles and phases are in degrees and rounded to the tenth place if necessary. Further information about each table can be found in its caption.

Table 2: TBM Two-angle Results ($\beta_{23} = 0$)

β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
168.3	351.0	0	[22] (5.0, 4.0, 1.3)	0.0048	4.2	9.0	0.2	0.0	180.0	-0.3	31.8	9.3	45.1	2.1×10^{-8}
348.3	351.0	0	(5.1, 3.1, 1.3)	0.0048	4.2	9.0	0.2	0.0	-180.0	179.8	31.9	9.3	44.8	4.5×10^{-7}
0.0	183.1	0	[21],[22] (4.7, 4.2, 2.0)	0.0047	9.1	3.1	0.2	180.0	0.0	-0.3	31.0	8.6	44.6	3.7×10^{-7}
170.5	354.8	0	(5.2, 5.0, 1.6)	0.0048	6.4	5.2	0.2	0.0	180.0	-0.3	36.1	8.2	44.8	1.1×10^{-7}
350.5	185.2	0	(5.2, 5.0, 1.6)	0.0048	6.4	5.2	0.2	180.0	0.0	-0.3	34.4	8.2	44.8	-5.9×10^{-8}
180.0	182.8	0	[22],[23] (4.8, 4.1, 2.0)	0.0048	9.0	2.7	0.3	-180.0	0.0	0.1	30.8	8.3	44.8	4.1×10^{-7}
10.5	174.8	0	[(22],[23],[32] (4.4, 3.8, 3.8)	0.0048	6.8	5.2	0.2	0.0	-180.0	0.7	36.4	8.5	44.8	2.1×10^{-6}
190.5	5.2	0	(4.4, 3.8, 3.8)	0.0048	6.8	5.2	0.2	-180.0	0.0	0.7	34.1	8.5	44.8	-2.2×10^{-6}
159.0	352.2	0	(5.1, 4.9, 1.3)	0.0047	3.9	7.8	0.1	0.0	-180.0	0.8	32.5	8.3	45.1	3.4×10^{-6}
164.6	186.0	0	[11],[13],[22] (5.1, 3.2, 1.5)	0.0047	5.5	6.0	0.1	180.0	0.0	-179.2	35.6	8.1	44.6	1.7×10^{-6}
164.6	354.0	0	(5.4, 5.2, 1.5)	0.0047	5.5	6.0	0.1	0.0	-180.0	0.8	34.4	8.2	44.9	4.4×10^{-6}
344.6	186.0	0	(5.4, 5.2, 1.5)	0.0047	5.5	6.0	0.1	-180.0	0.0	0.8	35.6	8.2	44.9	-4.5×10^{-6}
344.6	354.0	0	(5.1, 3.2, 1.5)	0.0047	5.5	6.0	0.1	0.0	-180.0	-179.2	34.9	8.2	44.6	-2.1×10^{-6}
174.7	178.8	0	[22],[23],[31] (5.1, 4.7, 2.6)	0.0048	9.6	3.6	0.3	179.9	0.0	0.1	31.0	9.3	44.7	-7.3×10^{-6}
343.2	178.9	0	[11],[22],[31] (5.4, 3.5, 2.7)	0.0048	9.6	3.2	0.1	180.0	0.0	-0.3	30.8	9.0	44.5	2.6×10^{-6}
164.3	353.0	0	[11],[13],[21],[22] (5.3, 5.9, 1.4)	0.0048	6.3	7.0	0.1	0.0	-180.0	-0.8	34.8	9.4	44.8	4.7×10^{-6}
171.7	358.3	0	[11],[21],[22],[31] (5.3, 4.4, 2.4)	0.0047	9.0	5.1	0.1	180.0	0.0	-0.3	32.5	10.0	44.5	-6.6×10^{-7}
351.7	358.3	0	(5.6, 3.8, 2.4)	0.0047	9.0	5.1	0.1	-180.0	0.0	179.8	32.4	9.9	44.2	-4.9×10^{-7}
175.5	1.6	0	[11],[12],[22],[31] (5.1, 4.0, 2.4)	0.0047	6.8	4.8	0.2	0.0	-180.0	0.8	36.7	8.2	44.8	1.1×10^{-7}
342.3	183.1	0	(5.5, 3.9, 2.0)	0.0048	4.1	9.3	0.1	0.0	-180.0	0.8	31.6	9.4	45.1	6.1×10^{-6}
355.5	178.4	0	(5.1, 4.0, 2.4)	0.0047	6.8	4.8	0.2	-180.0	0.0	0.8	33.8	8.2	44.8	-1.2×10^{-7}

(Table 1 continued on the next page)

Two-angle Results, TBM ($\beta_{23} = 0$) cont'd

β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
176.5	171.0	0	[12],[21],[22],[23] (4.6, 3.6, 1.3)	0.0046	4.2	9.0	0.3	0.0	180.0	-0.4	31.8	9.3	45.3	-9.3×10^{-7}
356.5	171.0	0	(5.1, 3.3, 1.3)	0.0046	4.1	9.0	0.3	0.0	-180.0	179.6	31.8	9.3	44.7	2.8×10^{-6}
349.4	4.0	0	[11],[13],[22],[23] (5.4, 5.1, 1.8)	0.0048	9.0	4.0	0.3	180.0	0.0	-0.4	31.7	9.2	44.7	2.5×10^{-8}
351.0	358.4	0	[11],[22],[23],[31] (5.7, 3.7, 2.4)	0.0048	6.9	4.7	0.3	-180.0	0.0	0.1	33.7	8.2	44.9	3.6×10^{-6}
162.0	182.0	0	[11],[12],[22],[23],[31] (5.2, 3.4, 2.3)	0.0048	6.6	5.9	0.3	0.0	180.0	-0.4	35.8	8.8	44.9	-4.5×10^{-6}
342.0	358.0	0	(5.2, 3.4, 2.3)	0.0048	6.6	5.9	0.3	180.0	0.0	-0.4	34.8	8.8	44.9	4.4×10^{-6}
166.6	177.9	0	[11],[21],[22],[23],[31] (6.2, 4.0, 2.2)	0.0049	6.2	6.3	0.3	-180.0	0.0	0.1	35.3	8.8	44.9	-5.2×10^{-6}
346.6	2.1	0	(6.2, 4.0, 2.2)	0.0049	6.2	6.3	0.3	0.0	-180.0	0.1	35.2	8.8	44.9	4.8×10^{-6}
339.4	3.8	0	[11],[12],[13],[21],[22] (5.2, 3.8, 1.8)	0.0046	9.2	3.8	0.1	-180.0	0.0	177.6	31.4	9.2	44.2	-8.6×10^{-6}
356.8	183.8	0	(4.8, 4.4, 1.8)	0.0046	8.2	3.8	0.1	180.0	0.0	-2.3	32.2	8.4	44.7	7.8×10^{-6}
171.7	4.5	0	[11],[13],[21],[22],[23] (5.5, 3.5, 1.7)	0.0047	9.0	4.5	0.3	-180.0	0.0	179.6	32.0	9.5	44.1	-2.8×10^{-6}
351.7	4.5	0	(5.1, 5.1, 1.7)	0.0047	9.0	4.5	0.3	180.0	0.0	-0.4	32.0	9.5	44.7	2.2×10^{-6}
339.4	4.9	0	[11],[12],[13],[22],[23] (5.2, 4.0, 1.7)	0.0047	9.2	4.7	0.3	-180.0	0.0	1.2	32.2	9.9	44.6	-9.4×10^{-6}

Table 2: Two-angle search results starting from an asymmetric $Y^{(-1/3)}$ with $\beta_{23} = 0$, TBM, $A = 0.77$, $(++)$ and 3σ range for neutrino mixing angles assumed. For each row there exists a corresponding solution with an additional GJ factor in the (32) entry.

Table 3: Two-angle Results, TBM ($\beta_{12} = 0$)

β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
★ 0.0	3.1	90.0	$\begin{matrix} [21],[23] \\ \begin{pmatrix} 4.7 & 3.8 & 3.7 \\ 4.2 & 2.2 & 2.8 \\ 2.0 & 0.0 & 4.9 \end{pmatrix} \end{matrix}$	0.0047	9.1	3.1	89.9	359.7	-0.3	-359.7	31.0	8.6	44.6	3.7×10^{-7}
0.0	177.2	269.0	$\begin{matrix} [22],[23] \\ \begin{pmatrix} 4.8 & 3.8 & 3.7 \\ 4.1 & 2.2 & 2.7 \\ 2.0 & 0.0 & 2.8 \end{pmatrix} \end{matrix}$	0.0048	9.0	2.7	88.7	0.0	0.0	0.0	31.0	8.4	45.8	4.1×10^{-7}
0.0	357.2	181.0	$\begin{pmatrix} 4.8 & 3.7 & 3.8 \\ 4.1 & 2.7 & 2.2 \\ 2.0 & 2.8 & 0.0 \end{pmatrix}$	0.0048	9.0	2.7	1.3	-180.0	0.0	0.0	31.0	8.4	45.8	4.1×10^{-7}
★ 0.0	183.1	0.0	$\begin{matrix} [21],[22] \\ \begin{pmatrix} 4.8 & 3.7 & 3.8 \\ 4.2 & 2.8 & 2.2 \\ 2.0 & 4.9 & 0.0 \end{pmatrix} \end{matrix}$	0.0047	9.1	3.1	0.2	180.0	0.0	-0.3	31.0	8.6	44.6	3.7×10^{-7}
0.0	177.2	270.0	$\begin{matrix} [22],[23],[33] \\ \begin{pmatrix} 4.8 & 3.8 & 3.7 \\ 4.1 & 2.2 & 2.8 \\ 2.0 & 0.0 & 4.9 \end{pmatrix} \end{matrix}$	0.0048	9.1	2.7	89.6	0.1	0.1	0.0	30.8	8.4	44.9	-5.6×10^{-8}
0.0	357.2	180.0	$\begin{matrix} [22],[23],[32] \\ \begin{pmatrix} 4.8 & 3.7 & 3.8 \\ 4.1 & 2.8 & 2.2 \\ 2.0 & 4.9 & 0.0 \end{pmatrix} \end{matrix}$	0.0048	9.1	2.7	0.5	-180.0	0.0	0.1	30.8	8.4	44.9	-5.6×10^{-8}

Table 3: Two-angle search results starting from an asymmetric $Y^{(-1/3)}$ with $\beta_{12} = 0$, TBM, $A = 0.77$, $(++)$ and 3σ range for neutrino mixing angles assumed. The row labelled by a “★” also has a corresponding solution with an additional GJ factor in the entry whose order of magnitude is 4.9.

Table 4: Two-angle Results, BM ($\beta_{23} = 0$)

β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
173.4	164.0	0	[22] (4.5, 3.2, 0.9)	0.0048	2.5	16.0	0.2	-180.0	180.0	179.8	31.8	9.5	46.3	1.0×10^{-6}
353.4	164.0	0	(4.8, 2.9, 0.9)	0.0048	2.4	16.0	0.2	180.0	-180.0	-0.2	31.8	9.5	46.6	-1.2×10^{-6}
187.0	176.9	0	[21],[22] (4.4, 3.7, 2.0)	0.0050	15.8	3.1	0.1	-180.0	180.0	179.7	31.6	8.9	44.2	-6.6×10^{-7}
201.0	357.9	0	[12],[22],[23] (4.1, 3.3, 2.2)	0.0048	13.7	2.0	0.3	-180.0	180.0	179.7	33.8	8.2	44.2	-3.4×10^{-7}
170.6	0.8	0	[11],[13],[21],[22] (5.8, 3.8, 2.9)	0.0047	12.5	0.8	0.2	-180.0	0.0	0.8	36.6	9.4	44.4	-1.3×10^{-6}
350.6	359.2	0	(5.8, 3.8, 2.9)	0.0047	12.5	0.8	0.2	180.0	-180.0	-179.2	35.5	8.3	44.3	2.4×10^{-6}
353.8	357.4	0	(5.3, 3.8, 2.1)	0.0048	14.9	2.6	0.2	180.0	-180.0	-179.2	32.6	8.6	44.2	2.1×10^{-6}
341.4	177.5	0	[11],[13],[22],[23] (5.1, 3.3, 2.1)	0.0049	14.7	2.4	0.3	-180.0	180.0	179.6	32.8	8.6	44.1	-2.4×10^{-6}
178.3	345.0	0	[12],[21],[22],[23] (4.4, 3.1, 0.9)	0.0048	2.5	15.0	0.3	-180.0	180.0	179.6	32.6	8.8	46.0	2.1×10^{-6}
358.3	345.0	0	(4.8, 3.0, 0.9)	0.0048	2.4	15.0	0.3	180.0	-180.0	-0.4	32.6	8.8	46.6	-3.3×10^{-6}
171.6	0.7	0	[11],[13],[21],[22],[31] (5.6, 3.9, 3.0)	0.0047	15.4	2.0	0.3	180.0	-180.0	-179.9	32.6	9.4	44.0	3.2×10^{-6}
350.4	359.8	0	(5.8, 3.9, 3.8)	0.0050	12.0	0.5	0.3	-180.0	0.0	0.1	36.8	8.9	44.6	4.3×10^{-6}
184.5	176.1	0	[11],[12],[13],[21],[22] (4.5, 3.8, 1.8)	0.0048	15.8	3.9	0.1	-180.0	180.0	177.7	31.0	8.4	44.4	-5.7×10^{-6}
5.0	181.3	0	[11],[12],[21],[22],[23],[31] (4.5, 4.1, 2.6)	0.0050	16.2	3.8	0.3	-180.0	180.0	179.6	30.8	8.7	44.1	-4.5×10^{-6}

Table 4: Two-angle search results starting from an asymmetric $Y^{(-1/3)}$, with $\beta_{23} = 0$, BM, $A = 0.77$, $(++)$ and 3σ range for neutrino mixing angles assumed. Each row, except for the last one, also has a corresponding solution with an additional GJ factor in the (32) entry.

Table 5: Two-angle Results, BM ($\beta_{12} = 0$)

β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
0	174.8	5.0	$\begin{matrix} [22],[23] \\ \begin{pmatrix} 4.9 & 3.8 & 3.8 \\ 3.8 & 2.9 & 2.2 \\ 1.6 & 1.6 & 0.0 \end{pmatrix} \end{matrix}$	0.0049	16.1	5.1	4.7	180.0	-180.0	0.0	30.7	9.0	49.3	1.9×10^{-6}
0	354.8	85.0	$\begin{matrix} \begin{pmatrix} 4.9 & 3.8 & 3.8 \\ 3.8 & 2.2 & 2.9 \\ 1.6 & 0.0 & 1.6 \end{pmatrix} \end{matrix}$	0.0049	16.1	5.1	85.3	0.0	180.0	0.0	30.7	9.0	49.3	1.9×10^{-6}
0	5.6	178.0	$\begin{matrix} [21],[22],[32] \\ \begin{pmatrix} 4.6 & 3.8 & 3.8 \\ 3.8 & 2.8 & 2.2 \\ 1.6 & 2.3 & 0.0 \end{pmatrix} \end{matrix}$	0.0046	16.0	5.5	6.0	180.0	-180.0	0.0	30.6	9.0	50.7	-2.4×10^{-6}
0	15.7	263.0	$\begin{matrix} [13],[21],[22],[23] \\ \begin{pmatrix} 4.3 & 3.8 & 3.8 \\ 3.1 & 2.2 & 3.0 \\ 0.9 & 0.0 & 1.4 \end{pmatrix} \end{matrix}$	0.0049	0.8	15.7	83.3	-1.5	180.0	0.0	32.1	9.0	52.8	5.6×10^{-5}
0	164.3	84.0	$\begin{matrix} \begin{pmatrix} 4.3 & 3.8 & 3.7 \\ 3.0 & 2.2 & 2.6 \\ 0.9 & 0.0 & 1.6 \end{pmatrix} \end{matrix}$	0.0048	0.7	15.7	83.7	1.6	180.0	0.0	32.2	9.2	52.4	4.6×10^{-5}
0	195.7	187.0	$\begin{matrix} [12],[21],[22],[23] \\ \begin{pmatrix} 4.3 & 3.8 & 3.8 \\ 3.1 & 3.0 & 2.2 \\ 0.9 & 1.4 & 0.0 \end{pmatrix} \end{matrix}$	0.0050	0.8	15.7	6.7	178.5	-180.0	0.0	32.1	9.0	52.8	-5.6×10^{-5}
0	344.3	6.0	$\begin{matrix} \begin{pmatrix} 4.3 & 3.7 & 3.8 \\ 3.0 & 2.6 & 2.2 \\ 0.9 & 1.6 & 0.0 \end{pmatrix} \end{matrix}$	0.0048	0.7	15.7	83.7	1.6	180.0	0.0	32.2	9.2	52.4	4.6×10^{-5}

Table 5: Two-angle search results starting from an asymmetric $Y^{(-1/3)}$, with $\beta_{12} = 0$, BM, $A = 0.77$, $(++)$ and 3σ range for neutrino mixing angles assumed. In this case there are no additional solutions found by the addition of GJ factors into small entries.

Table 6: Two-angle Results, BM ($\beta_{13} = 0$)

β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
123.2	0	172.1	$\begin{matrix} [12],[13],[22] \\ \left(\begin{smallmatrix} 4.0 & 3.9 & 3.8 \\ 2.9 & 2.9 & 2.2 \\ 5.0 & 1.3 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0048	16.5	0.0	7.8	180.0	-9.5	-180.0	31.7	9.9	36.1	9.2×10^{-6}
316.0	0	185.8	$\begin{matrix} \left(\begin{smallmatrix} 4.2 & 3.8 & 3.8 \\ 3.0 & 2.8 & 2.2 \\ 5.2 & 1.5 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0048	12.9	0.0	5.7	-180.0	170.2	0.1	36.7	10.0	50.0	-3.8×10^{-6}
136.0	0	84.2	$\begin{matrix} [12],[13],[23] \\ \left(\begin{smallmatrix} 4.2 & 3.8 & 3.8 \\ 3.0 & 2.2 & 2.8 \\ 5.2 & 0.0 & 1.5 \end{smallmatrix} \right) \end{matrix}$	0.0048	12.9	0.0	84.3	0.1	170.3	-0.1	36.7	10.0	50.0	-3.8×10^{-6}
303.2	0	97.9	$\begin{matrix} \left(\begin{smallmatrix} 4.0 & 3.8 & 3.9 \\ 2.9 & 2.2 & 2.9 \\ 5.0 & 0.0 & 1.3 \end{smallmatrix} \right) \end{matrix}$	0.0048	16.5	0.0	82.2	-180.0	170.5	180.0	31.7	9.9	36.1	9.2×10^{-6}
134.3	0	88.2	$\begin{matrix} [13],[22],[23],[33] \\ \left(\begin{smallmatrix} 4.1 & 3.8 & 3.8 \\ 3.0 & 2.2 & 2.9 \\ 5.1 & 0.0 & 2.3 \end{smallmatrix} \right) \end{matrix}$	0.0049	13.8	0.1	85.0	-180.0	0.3	180.0	34.3	8.7	39.2	2.6×10^{-6}
316.5	0	91.9	$\begin{matrix} \left(\begin{smallmatrix} 4.2 & 3.8 & 3.8 \\ 3.0 & 2.2 & 2.9 \\ 5.2 & 0.0 & 2.3 \end{smallmatrix} \right) \end{matrix}$	0.0046	12.7	0.1	5.3	0.0	0.3	0.0	36.9	9.8	49.6	-1.8×10^{-6}
314.3	0	181.8	$\begin{matrix} [12],[22],[23],[32] \\ \left(\begin{smallmatrix} 4.1 & 3.8 & 3.8 \\ 3.0 & 2.9 & 2.2 \\ 5.1 & 2.3 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0049	13.8	0.1	5.0	180.0	-179.7	-180.0	34.3	8.7	39.2	2.6×10^{-6}
136.5	0	178.1	$\begin{matrix} \left(\begin{smallmatrix} 4.2 & 3.8 & 3.8 \\ 3.0 & 2.9 & 2.2 \\ 5.2 & 2.3 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0046	12.7	0.1	5.3	-180.0	0.3	0.0	36.9	9.8	49.6	-1.8×10^{-6}

Table 6: Two-angle search results starting from an asymmetric $Y^{(-1/3)}$, with $\beta_{13} = 0$, BM, $A = 0.77$, $(++)$ and 3σ range for neutrino mixing angles assumed. Each row also has a corresponding solution with an additional GJ factor in the entry whose order of magnitude is around or above 5.0.

Solutions with three angles in \mathcal{V}

We organize the three-angle solutions according to the cases of TBM or BM, β_{23} close to the x-axis or the y-axis, and normal hierarchy(NH) or inverted hierarchy(IH) solutions. 1σ ranges for neutrino mixing angles are used.

Table 7: Full three-angle NH solutions (also serve as part of the three-angle IH solutions) for TBM and β_{23} close to the x-axis, with $A = 0.77$, (+,+) and 1σ ranges for neutrino mixing angles assumed.

GJ Entries	β_{12}	β_{13}	β_{23}
[22],[32]	168.5	188.1	181.0
	348.5	351.9	1.0
	168.3	351.6	1.8
	348.3	188.4	181.8
[21],[22]	169.2	353.9	175.8
	349.2	186.1	355.8
[12],[21],[22]	7.9	174.7	176.6
	187.9	5.3	356.6
	12.4	173.7	176.7
	192.4	6.3	356.2
[22],[23],[32]	154.6	174.3	181.9
	334.6	5.7	1.8
	163.4	175.2	181.5
	343.4	4.8	1.5
[11],[13],[22]	163.8	351.5	355.3
	343.8	188.5	175.3
[11],[22],[31]	170.1	357.3	176.7
	350.1	182.7	356.7
[21],[22],[32]	171.0	354.7	181.2
	351.0	185.3	1.2
[12],[21],[22],[32]	11.2	174.5	181.4
	191.2	5.5	1.4
[11],[13],[22],[32]	164.9	187.7	181.3
	344.9	352.3	1.3
	167.1	351.1	1.9
	344.6	187.0	181.3
[11],[13],[21],[22]	163.4	352.4	355.3
	343.4	187.6	175.3

[11],[21],[22],[31]	171.4	358.0	356.0
	351.4	181.6	176.5
[11],[12],[22],[31]	173.5	1.8	177.6
	353.5	178.1	356.4
[12],[21],[22],[23]	177.0	171.6	355.0
	357.0	8.4	175.0
[11],[22],[31],[32]	347.9	182.7	1.0
	167.9	357.3	181.0
[12],[21],[22],[23],[31]	11.0	358.2	355.0
	191.0	181.8	175.0
[11],[12],[22],[23],[31]	161.1	182.1	174.4
	341.1	357.9	354.4
[11],[21],[22],[23],[31]	166.6	177.9	354.3
	346.6	2.1	174.3
[11],[22],[23],[31],[32]	168.6	181.9	180.9
	348.6	358.0	1.4
	171.5	181.9	181.3
	351.5	358.1	1.3
[11],[12],[13],[21],[22]	173.3	354.8	176.5
	353.3	185.1	356.8
[11],[12],[22],[31],[32]	174.4	1.9	181.2
	354.4	178.1	1.2
[12],[21],[22],[23],[32]	175.6	171.9	1.1
	355.6	8.1	181.1
[11],[12],[13],[22],[23],[32]	157.4	174.5	181.7
[11],[13],[21],[22],[23],[32]	171.1	174.1	181.4
	351.1	4.9	1.4

Table 8: Other three-angle IH solutions for TBM and β_{23} close to the x-axis, with $A = 0.77$, $(+,+)$ and 1σ ranges for neutrino mixing angles assumed.

GJ Entries	β_{12}	β_{13}	β_{23}
[22]	168.3	187.9	179.6
	168.1	352.1	359.7
	348.1	187.9	179.7
	348.3	352.0	359.6
[21],[22]	155.3	175.1	185.4
	335.3	4.9	5.4
	351.0	185.2	0.0
[22],[32]	168.2	352.3	359.0
	347.8	352.4	358.2
[11],[13],[22]	164.6	186.6	179.9
	344.6	352.4	0.0
[12],[21],[22]	186.5	4.3	0.1
[11],[22],[32]	350.6	185.2	358.5
[11],[22],[31],[32]	167.0	352.1	358.3
	344.0	353.2	357.8
[11],[13],[22],[23]	169.2	176.3	184.9
	349.2	3.7	4.9
[11],[21],[22],[31]	171.6	181.3	183
	351.6	358.3	4.1
[12],[21],[22],[23]	175.4	7.1	186.5
	355.4	172.4	6.2
	356.5	172.1	359.6
[12],[21],[22],[32]	185.5	3.9	357.8

[11],[12],[13],[21],[22]	157.5	175.9	184.0
	337.5	4.1	4.0
[11],[12],[22],[23],[31]	160.2	182.1	5.7
	176.6	178.9	6.4
	340.2	357.9	185.7
	356.6	1.1	186.4
[11],[13],[21],[22],[32]	164.4	353.1	0.0
	166.6	353.7	357.8
	344.4	186.9	180
	346.6	186.3	177.8
[11],[21],[22],[23],[31]	166.6	177.9	182.7
	346.6	2.1	2.7
[12],[21],[22],[23],[32]	176.0	7.7	179.1
	356.0	172.3	359.1
[11],[12],[13],[21],[22],[32]	159.0	175.7	178.8
	174.0	353.5	179
	338.7	4.3	358.8
	354.0	184.4	358.0
[11],[12],[22],[23],[31],[32]	161.7	182.1	180.2
	341.7	357.9	0.2
[11],[13],[21],[22],[23],[32]	171.3	3.8	358.6
	351.3	176.2	178.6

Table 9: Full three-angle NH solutions (also serve as part of three-angle IH solutions) for TBM and β_{23} close to the y-axis, with $A = 0.77$, $(+,+)$ and 1σ ranges for neutrino mixing angles assumed.

GJ Entries	β_{12}	β_{13}	β_{23}
[23],[33]	168.5	8.1	269.0
	168.3	171.6	88.1
[21],[23]	169.2	173.9	274.1
	349.2	6.1	94.1
[13],[21],[23]	6.7	355.3	272.4
	186.7	184.7	92.4
[22],[23],[33]	154.6	354.3	268.1
	334.6	185.7	88.1
	343.0	184.6	89.1
[11],[12],[23]	163.8	171.5	94.4
	343.8	8.0	274.6
[11],[23],[31]	170.1	177.3	273.2
	350.1	2.7	93.2
[21],[23],[33]	171.0	174.7	268.8
	351.0	5.3	88.8
[13],[21],[23],[33]	7.2	355.4	269.3
	11.0	354.6	269.1
	187.2	184.6	89.3
[11],[12],[23],[33]	164.8	6.8	269.2
	164.5	173.1	88.9
	344.5	6.7	269.2
[11],[23],[31],[33]	167.9	177.3	268.1
	347.9	2.7	88.1
[11],[21],[23],[31]	171.4	178.0	93.7
	351.4	1.6	273.2

[13],[21],[22],[23]	176.8	351.7	93.4
	356.8	188.0	273.0
[11],[12],[21],[23]	343.4	7.6	274.2
[11],[13],[23],[31]	353.5	358.1	93.3
[13],[21],[22],[23],[31]	11.0	178.2	94.3
	191.0	1.8	274.3
[11],[13],[22],[23],[31]	161.1	2.1	275.4
	341.1	177.9	95.4
[11],[21],[22],[23],[31]	166.6	357.9	93.4
	346.6	182.1	273.4
[11],[22],[23],[31],[33]	168.6	1.9	269.0
	348.6	178.0	88.5
[11],[12],[13],[21],[23]	173.3	174.8	273.5
	353.3	5.1	93.0
[11],[13],[23],[31],[33]	174.4	181.9	268.4
	354.4	358.1	88.3
[13],[21],[22],[23],[33]	175.6	351.9	88.9
	355.6	188.1	268.9
[11],[12],[13],[22],[23],[33]	157.4	354.5	268.3
	337.4	185.5	88.3
[11],[12],[21],[22],[23],[33]	171.1	354.1	268.6
	351.1	184.9	88.6

Table 10: Other three-angle IH solutions for TBM and β_{23} close to the y-axis, with $A = 0.77$, $(+,+)$ and 1σ ranges for neutrino mixing angles assumed.

GJ Entries	β_{12}	β_{13}	β_{23}
[23]	168.1	172.1	90.3
	168.3	7.9	270.0
	348.1	7.9	270.3
	348.3	172.0	90.2
[21],[23]	155.3	355.1	264.5
	170.0	174.3	272.2
	335.3	184.9	84.5
	350.0	5.7	92.2
[23],[33]	167.8	7.4	271.8
	347.8	172.4	91.8
[13],[21],[23]	6.5	355.6	270.6
	10.5	354.8	270.0
	186.5	184.3	89.9
	190.5	185.2	90.0
[22],[23],[33]	155.5	354.8	269.2
	162.9	355.7	269.8
	335.5	185.2	89.2
	342.9	184.3	89.8
[11],[12],[23]	164.6	6.6	269.8
	167.1	171.8	90.3
	344.1	8.1	272.6
	344.6	172.4	89.9
	347.1	171.6	90.1
[11],[23],[31]	170.2	177.3	272.0
	350.2	2.6	91.0
[21],[23],[33]	170.6	174.8	271.3
	350.6	5.2	91.3
[13],[21],[23],[33]	5.5	356.1	272.1
	185.5	183.9	92.1
[11],[12],[23],[33]	164.0	6.8	272.2
	164.5	173.2	89.3
	344.5	6.7	269.3
	344.1	173.0	91.7
[11],[23],[31],[33]	167.9	177.3	269.2
[11],[12],[21],[23]	163.8	172.7	92.4
	171.6	1.3	266.4
	343.8	7.3	272.4
[11],[12],[22],[23]	169.2	356.3	264.3
[11],[21],[23],[31]	171.5	178.3	92.0

[13],[21],[22],[23]	175.4	187.1	263.3
	176.6	351.9	91.5
	355.4	352.4	83.7
	356.6	187.9	271.7
[11],[13],[23],[31]	353.6	358.2	92.0
[11],[12],[13],[21],[23]	157.5	355.9	265.8
	173.5	175.1	271.9
	337.5	184.1	85.8
	353.5	4.9	91.9
[11],[13],[22],[23],[31]	160.2	2.1	84.2
	168.7	1.8	269.6
	341.9	178.0	91.6
	340.2	177.9	264.2
	356.6	181.1	263.5
[11],[12],[21],[23],[33]	164.4	173.1	89.7
	344.4	6.9	269.7
[11],[21],[22],[31],[23]	166.6	357.9	263.7
	346.6	182.1	83.7
[11],[13],[23],[31],[33]	176.0	181.8	269.3
	356.0	358.2	89.3
[13],[21],[22],[23],[33]	176.0	187.7	270.9
	355.7	188.1	269.0
[11],[22],[23],[31],[33]	348.7	178.1	89.2
[11],[12],[13],[22],[23],[33]	157.7	354.9	269.2
	337.7	185.1	89.2
[11],[12],[13],[21],[23],[33]	159.0	355.7	271.1
	174.0	175.5	270.8
	339.0	184.3	91.1
	161.7	2.1	269.8
[11],[13],[22],[23],[31],[33]	161.7	2.1	269.8
	341.7	177.9	89.8
[11],[12],[21],[22],[23],[33]	171.3	183.8	91.4
	171.4	354.8	269.2
	351.4	184.9	89.2
	351.3	356.2	271.4

Table 11: Full three-angle NH solutions (also serve as part of three-angle IH solutions) for BM and β_{23} close to the x-axis, with $A = 0.77$, $(+,+)$ and 1σ ranges for neutrino mixing angles assumed. Note that solutions with “ \star ” ’s belong to NH solutions only.

GJ Entries	β_{12}	β_{13}	β_{23}
[22]	174.7	14.1	175.8
	354.7	165.9	355.8
[22],[32]	172.7	14.5	182.0
	173.0	165.4	1.8
	353.0	14.4	181.9
	352.8	165.5	2.1
[12],[22],[23]	24.4	182.0	177.5
	204.6	358.0	357.3
[12],[13],[22]	\star 131.3	0.7	174.9
	\star 311.3	179.3	354.9
[12],[13],[22],[31]	131.8	359.9	175.2
	311.8	180.1	355.2
[12],[22],[23],[32]	\star 133.0	180.0	1.8
	\star 313.0	0.0	181.8
[12],[21],[22],[23]	176.5	194.4	173.5
	176.6	345.4	353.0
	356.6	194.4	173.0
	356.5	345.6	353.5
[11],[13],[21],[22],[32]	171.0	181.5	180.9
	351.0	358.5	0.9

Table 12: Other three-angle IH solutions for BM and β_{23} close to the x-axis, with $A = 0.77$, $(+,+)$ and 1σ ranges for neutrino mixing angles assumed.

GJ Entries	β_{12}	β_{13}	β_{23}
[22]	172.4	164.7	4.3
	352.4	15.3	184.3
	353.2	165.2	0.5
[22],[32]	173.4	165.1	359.6
	353.4	165.1	359.8
[21],[22],[32]	3.6	3.6	177.8
	183.6	176.4	357.8
[12],[22],[23]	199.4	357.9	4.0
[12],[21],[22],[23]	359.5	194.9	185.2
	178.4	194.6	180.3
	178.5	345.1	1.0
	358.4	345.1	0.3
[12],[22],[23],[31]	21.6	179.0	184.4
	201.6	1.0	4.4
[12],[22],[23],[32]	21.4	183.3	178.6
	201.4	356.7	358.6
[11],[22],[23],[31]	161.4	358.8	184.0
	341.4	181.2	4.0
[11],[13],[21],[22]	170.8	180.9	179.0
	350.8	358.9	359.9

[13],[21],[22],[23],[31]	3.0	181.3	5.7
	183.0	358.7	185.7
[12],[22],[23],[31],[32]	28.5	179.0	179.0
	208.5	1.0	359.0
[11],[13],[22],[23],[32]	161.4	2.5	180.0
	341.4	177.3	359.8
[11],[21],[22],[23],[31]	173.4	0.7	2.7
	353.4	179.3	182.7
[11],[13],[21],[22],[32]	173.5	182.0	178.7
	350.9	358.8	0.0
[12],[21],[22],[23],[32]	178.3	194.7	179.8
	177.8	344.7	358.7
	357.8	194.5	178.7
	358.3	345.1	359.7
[11],[21],[22],[23],[31],[32]	171.0	0.5	359.7
	351.0	179.3	178.9

Table 13: Full three-angle NH solutions (also serve as part of three-angle IH solutions) for BM and β_{23} close to the y-axis, with $A = 0.77$, $(+,+)$ and 1σ ranges for neutrino mixing angles assumed.

GJ Entries	β_{12}	β_{13}	β_{23}
[23]	174.5	194.0	274.0
	354.5	346.0	94.0
[23],[33]	172.6	194.4	267.6
	172.9	345.4	87.6
	352.9	194.3	267.6
	352.6	345.4	87.6
[22],[13],[23]	24.3	2.0	272.3
	131.3	180.7	275.1
	204.3	178.0	92.3
	311.3	359.2	95.1
[13],[12],[31],[23]	131.8	179.9	274.8
	311.8	0.1	94.8
[22],[13],[23],[33]	133.0	0.0	88.2
	313.0	180.0	268.2
[21],[22],[13],[23]	176.5	14.4	276.5
	356.5	165.6	96.5
[13],[22],[31],[23],[33]	133.0	0.1	88.4
	313.0	179.9	268.4
[11],[12],[21],[23],[33]	170.9	1.4	269.3
	350.9	178.6	89.3

Table 14: Other three-angle IH solutions for BM and β_{23} close to the y-axis, with $A = 0.77$, $(+,+)$ and 1σ ranges for neutrino mixing angles assumed.

GJ Entries	β_{12}	β_{13}	β_{23}
[23]	172.4	344.7	85.7
	173.2	194.8	269.5
	352.4	195.3	265.7
	353.2	345.2	89.3
[23],[33]	172.9	194.5	268.7
	173.1	345.3	88.7
	353.1	194.5	268.7
	352.9	345.4	88.7
[21],[23],[33]	3.6	183.6	272.2
	183.6	356.4	92.2
[13],[22],[23]	19.4	2.1	265.9
	199.4	177.9	85.9
[13],[22],[23],[31]	21.6	359.0	265.6
	201.6	181.0	85.6
[13],[22],[23],[33]	23.0	2.1	270.0
	201.4	176.7	91.4
[11],[22],[23],[31]	161.4	178.8	262.9
	341.4	1.2	82.9
[11],[12],[22],[23]	162.0	182.6	267.1
	342.0	357.4	87.1
[11],[12],[21],[23]	170.8	0.9	271.0
	350.8	178.9	90.0

[13],[21],[22],[23]	177.2	14.5	274.0
	177.4	165.3	93.6
	357.2	165.5	94.0
	357.4	14.7	273.6
[12],[21],[22],[23],[31]	3.0	1.3	84.1
	183.0	178.7	264.1
[13],[22],[23],[31],[33]	28.5	359.0	271.0
	208.5	181.0	91.0
[11],[12],[22],[23],[33]	170.8	1.1	269.9
	350.8	178.8	89.6
[11],[21],[22],[23],[31]	173.4	180.7	86.0
	353.4	359.3	266.0
[13],[21],[22],[23],[33]	178.3	14.7	270.1
	177.8	164.7	91.3
	357.8	14.5	271.3
	358.3	165.1	90.1
[11],[21],[22],[23],[31],[33]	171.0	180.5	90.2
	351.0	359.3	271.0

B Search Results: Almost Symmetric $Y^{(-1/3)}$

As in Appendix A, we list the results of our search in table form. Table 15 gives our result for the TBM case, while Table 16 shows the result for BM mixing.

The layout of the tables is similar to that used in Appendix A, where we now study the deviation γ_{ij} from the symmetric matrix. In addition, we have also varied the Wolfenstein parameter A , and these two parameters may be found in the second column. In the third column we list the corresponding β_{ij} for a given solution matrix, to allow comparison with the three-angle results starting from an asymmetric matrix.

As for these examples all of the entries of $Y^{(-1/3)}$ tend to vary, we list their orders of magnitude as done for the BM case in column 4. The rest of the columns give the various outputs defined earlier.

Table 15: One-angle Almost-Symmetric Results TBM

	A	γ_{13}	β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
(++)	0.76	173.2	13.4	6.7	2.3	$\begin{pmatrix} [12],[21],[22] \\ 4.4 & 3.8 & 3.8 \\ 4.7 & 2.7 & 2.2 \\ 1.4 & 2.2 & 0.0 \end{pmatrix}$	0.0047	7.3	6.7	2.1	-356.4	178.6	180.0	36.1	9.8	42.4	-1.6×10^{-3}
	0.71	359.0	13.2	0.9	2.1	$\begin{pmatrix} [12],[22],[23],[31] \\ 4.3 & 3.8 & 3.9 \\ 3.9 & 2.7 & 2.2 \\ 2.8 & 2.2 & 0.0 \end{pmatrix}$	0.0049	9.5	2.7	2.4	-174.3	350.0	0.0	30.7	8.8	46.8	3.9×10^{-3}
	0.76	357.9	13.2	2.0	2.2	$\begin{pmatrix} [12],[21],[22],[23],[31] \\ 4.3 & 3.8 & 3.8 \\ 4.1 & 2.7 & 2.2 \\ 2.3 & 2.2 & 0.0 \end{pmatrix}$	0.0048	6.7	6.0	2.5	-165.8	-4.8	0.0	35.3	8.9	47.2	6.4×10^{-3}
(+-)	0.60	7.1	12.9	7.2	1.8	$\begin{pmatrix} [11],[13],[22],[32] \\ 5.4 & 3.7 & 4.0 \\ 6.4 & 2.8 & 2.3 \\ 1.4 & 2.3 & 0.0 \end{pmatrix}$	0.0047	4.0	7.2	5.4	-179.8	1.1	180.0	36.7	8.1	39.5	-3.6×10^{-4}
	0.60	170.9	13.4	9.0	1.8	$\begin{pmatrix} 5.2 & 3.7 & 4.0 \\ 3.2 & 2.8 & 2.4 \\ 1.3 & 2.3 & 0.0 \end{pmatrix}$	0.0047	5.5	9.0	5.4	0.2	179.1	0.0	31.9	10.0	50.2	-4.2×10^{-4}
	0.60	7.2	12.9	7.3	1.8	$\begin{pmatrix} [11],[13],[21],[22],[32] \\ 5.4 & 3.7 & 4.0 \\ 6.4 & 2.8 & 2.3 \\ 1.4 & 2.3 & 0.0 \end{pmatrix}$	0.0049	4.1	7.3	5.4	-184.4	1.1	180.0	36.7	8.2	39.5	-1.2×10^{-3}
	0.67	182.1	13.0	2.2	2.0	$\begin{pmatrix} [11],[21],[22],[23],[31] \\ 6.2 & 3.7 & 3.9 \\ 4.0 & 2.8 & 2.3 \\ 2.2 & 2.3 & 0.0 \end{pmatrix}$	0.0046	4.9	6.6	1.7	-14.7	-176.2	180.0	34.2	8.1	43.0	4.3×10^{-3}

(Table continued on the next page)

One-angle Almost-Symmetric Case TBM, cont'd

	A	γ_{13}	β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
(-+)	0.61	172.1	13.4	7.8	1.8	$\begin{matrix} [11],[13],[21],[22] \\ \left(\begin{smallmatrix} 5.4 & 3.7 & 4.0 \\ 5.6 & 2.7 & 2.3 \\ 1.3 & 2.4 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0048	5.1	7.8	1.7	3.5	1.0	180.0	33.7	9.2	43.1	-1.1×10^{-3}
	0.61	172.1	13.4	7.8	1.8	$\begin{matrix} [11],[13],[21],[22],[32] \\ \left(\begin{smallmatrix} 5.4 & 3.7 & 4.0 \\ 5.6 & 2.7 & 2.3 \\ 1.3 & 2.4 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0048	6.5	7.8	5.4	3.1	-181.0	0.0	33.5	10.0	50.2	-1.2×10^{-3}
	0.64	357.5	13.2	2.4	1.9	$\begin{matrix} [11],[21],[22],[23],[31] \\ \left(\begin{smallmatrix} 6.2 & 3.7 & 3.9 \\ 4.0 & 2.7 & 2.3 \\ 2.1 & 2.3 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0049	5.5	7.2	2.1	-168.2	-3.3	0.0	36.8	8.9	46.9	4.6×10^{-3}
	0.6	357.1	13.2	2.8	1.8	$\begin{matrix} [11],[21],[22],[23],[31],[32] \\ \left(\begin{smallmatrix} 6.1 & 3.7 & 4.0 \\ 4.0 & 2.7 & 2.4 \\ 2.0 & 2.4 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0048	5.0	8.4	4.9	-171.2	-2.7	180.0	36.8	9.6	39.8	3.2×10^{-3}
(-)	0.8	5.9	12.9	6.0	2.4	$\begin{matrix} [12],[21],[22],[32] \\ \left(\begin{smallmatrix} 4.4 & 3.8 & 3.7 \\ 4.8 & 2.8 & 2.1 \\ 1.5 & 2.1 & 0.0 \end{smallmatrix} \right) \end{matrix}$	0.0048	7.3	6.0	7.1	-184.3	1.7	180.0	33.1	9.2	37.3	-1.9×10^{-3}

Table 15: One-angle almost-symmetric case with TBM, non-zero γ_{13} and 3σ range for neutrino mixing angles.

Table 16: One-angle almost-symmetric case with BM, non-zero γ_{12} and 3σ range for neutrino mixing angles.

	A	γ_{12}	β_{12}	β_{13}	β_{23}	GJ Entries	m_e/m_μ	θ_{12}^e	θ_{13}^e	θ_{23}^e	δ_{12}^e	δ_{13}^e	δ_{23}^e	θ_{12}	θ_{13}	θ_{23}	J_{MNSP}
(++)	0.7	158.3	8.6	0.6	1.7	$\begin{matrix} [11],[21],[22],[23],[31] \\ \begin{pmatrix} 5.6 & 3.7 & 4.0 \\ 3.9 & 2.7 & 2.3 \\ 3.1 & 2.4 & 0.0 \end{pmatrix} \end{matrix}$	0.0048	14.8	2.0	2.2	-175.3	167.4	178.5	32.8	8.6	42.1	2.2×10^{-3}
	0.61	337.3	9.6	0.6	1.7	$\begin{matrix} [11],[21],[22],[23],[31],[32] \\ \begin{pmatrix} 5.8 & 3.7 & 4.0 \\ 3.8 & 2.7 & 2.3 \\ 3.1 & 2.4 & 0.0 \end{pmatrix} \end{matrix}$	0.0047	12.0	1.8	4.7	-176.2	-11.9	178.1	36.9	9.1	39.5	3.2×10^{-3}

C Analysis of the Results

Although our solutions were found numerically, we would like to try and understand them analytically so as to answer two questions: How are the particular mixing angles in \mathcal{V} singled out? What are the effects of the insertions of GJ factors? We investigate these questions in two steps: first, find out what the needed mixing angles in \mathcal{U}_{-1} are, given a particular form of $\mathcal{U}_{\text{seesaw}}$; second, connect them with the input angles in \mathcal{V} through the influences of GJ factors.

Mixing angles in \mathcal{U}_{-1}

We start with a useful parametrization of \mathcal{U}_{-1} and $\mathcal{U}_{\text{MNSP}}$ that may be used to calculate the deviations of the mixing angles and phase in $\mathcal{U}_{\text{MNSP}}$ from their values in $\mathcal{U}_{\text{seesaw}}$. Following [15], an Iwasawa decomposition of the unitary matrices \mathcal{U}_{-1} and $\mathcal{U}_{\text{seesaw}}$ is given by,

$$\begin{aligned}\mathcal{U}_{-1} &= P_L^e R_{23}^e U_{13}^{e'} R_{12}^e P_R^e, \\ \mathcal{U}_{\text{seesaw}} &= P_L^\nu R_{23}^\nu U_{13}^{\nu'} R_{12}^\nu P_R^\nu,\end{aligned}\tag{C.1}$$

where P_L and P_R are phase matrices, R_{ij} is a rotation matrix, and U_{ij} are unitary matrices defined by,

$$\begin{aligned}P_L &= \begin{pmatrix} e^{i\varphi_1} & & \\ & e^{i\varphi_2} & \\ & & e^{i\varphi_3} \end{pmatrix}, & P_R &= \begin{pmatrix} e^{i\beta_1} & & \\ & e^{i\beta_2} & \\ & & 1 \end{pmatrix}, \\ R_{12} &= \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & U_{12} &= \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{12}} & 0 \\ -s_{12}e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ R_{13} &= \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, & U_{13} &= \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix}, \\ R_{23} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, & U_{23} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\delta_{23}} \\ 0 & -s_{23}e^{i\delta_{23}} & c_{23} \end{pmatrix}.\end{aligned}$$

Inserting this parametrization into Eq.(3), the MNSP matrix may then be written as,

$$\begin{aligned}\mathcal{U}_{\text{MNSP}} &= \mathcal{U}_{-1}^\dagger \mathcal{U}_{\text{seesaw}} \\ &= (P_R^{e\dagger} P_L^{e\dagger} P_L^\nu P_R^\nu) U_{12}^{e\dagger} U_{13}^{e\dagger} U_{23}^{e\dagger} U_{23}^\nu U_{13}^\nu U_{12}^\nu,\end{aligned}\tag{C.2}$$

by commuting the various phase matrices to the left. This shift of the phase matrices to the left has the effect of both placing phases in the original rotation

matrices of Eq.(C.1) and also of shifting the phases from their original values in \mathcal{U}'_{ij} . The relations between the old phases of Eq.(C.1) and those of Eq.(C.2) are given by,

$$\begin{aligned}
\delta_{12}^\nu &= \beta_1^\nu - \beta_2^\nu, \\
\delta_{13}^\nu &= \delta_{13}' + \beta_1^\nu, \\
\delta_{23}^\nu &= \beta_2^\nu, \\
\delta_{12}^e &= [(\varphi_2^\nu + \beta_2^\nu) - \varphi_2^e] - [(\varphi_1^\nu + \beta_1^\nu) - \varphi_1^e], \\
\delta_{13}^e &= \delta_{13}' - (\varphi_3^\nu - \varphi_3^e) + [(\varphi_1^\nu + \beta_1^\nu) - \varphi_1^e], \\
\delta_{23}^e &= (\varphi_3^\nu - \varphi_3^e) - [(\varphi_2^\nu + \beta_2^\nu) - \varphi_2^e].
\end{aligned}$$

Similarly, we decompose the MNSP matrix as,

$$\mathcal{U}_{\text{MNSP}} = P_L U_{23} U_{13} U_{12}. \quad (\text{C.3})$$

The relations between the mixing angles in the MNSP matrix and those in \mathcal{U}_{-1} and $\mathcal{U}_{\text{seesaw}}$ can then be obtained by studying the expansion of the right hand side of Eq.(C.2) in terms of small angles, if any.

In our case, fortunately, most angles are small, because the seesaw matrices are designed to account for the large angles in the data. Thus the expansion parameters are θ_{13}^ν and the θ_{ij}^e , although the search results indicate that θ_{23}^e can sometimes be quite close to the y-axis. We therefore consider these two distinct cases separately:

- $|s_{13}'|, |s_{12}^e|, |s_{13}^e|, |s_{23}^e| \ll 1$:

The first order relations between the mixing angles in the MNSP matrix and those in \mathcal{U}_{-1} and $\mathcal{U}_{\text{seesaw}}$ simplify to,

$$\begin{aligned}
s_{23} e^{-i\delta_{23}} &\approx s_{23}' e^{-i\delta_{23}^\nu} - \theta_{23}^e c_{23}' e^{-i\delta_{23}^e}, \\
\theta_{13} e^{-i\delta_{13}} &\approx \theta_{13}' e^{-i\delta_{13}^\nu} - \theta_{13}^e c_{23}' e^{-i\delta_{13}^e} - \theta_{12}^e s_{23}' e^{-i(\delta_{23}^\nu - \delta_{12}^e)}, \\
s_{12} e^{-i\delta_{12}} &\approx s_{12}' e^{-i\delta_{12}^\nu} + \theta_{13}^e c_{12}' s_{23}' e^{i(\delta_{23}^\nu - \delta_{13}^e)} - \theta_{12}^e c_{23}' c_{12}' e^{-i\delta_{12}^e}, \quad (\text{C.4})
\end{aligned}$$

with $\delta^{\text{MNSP}} = \delta_{13} - \delta_{12} - \delta_{23}$. The above formulae will allow us to easily identify the sources of corrections to the mixing angles in $\mathcal{U}_{\text{seesaw}}$.

Further simplifications occur when restricting ourselves to TBM- and BM-compatible seesaw matrices of the form Eq.(10), whose Iwasawa decomposition gives

$$\begin{aligned}
\theta_{13}^\nu &= 0, \quad \theta_{23}^\nu = 45^\circ, \\
\varphi_1^\nu &= 0, \quad \varphi_2^\nu = 180^\circ, \quad \varphi_3^\nu = 0, \\
\beta_1^\nu &= 0, \quad \beta_2^\nu = 180^\circ.
\end{aligned} \quad (\text{C.5})$$

The relations in Eq.(C.4) reduce to

$$\begin{aligned} -s_{23}e^{-i\delta_{23}} &= \sin\left(\frac{\pi}{4} + \delta\theta_{23}\right), \\ \theta_{13}e^{-i\delta_{13}} &= \delta\theta_{13}, \\ -s_{12}e^{-i\delta_{12}} &= \sin(\theta_{12}^\nu + \delta\theta_{12}), \end{aligned} \quad (\text{C.6})$$

where

$$\begin{aligned} \delta\theta_{23} &\approx \theta_{23}^e e^{-i\delta_{23}^e}, \\ \delta\theta_{13} &\approx \frac{1}{\sqrt{2}} \left(\theta_{12}^e e^{-i\delta_{12}^e} - \theta_{13}^e e^{-i\delta_{13}^e} \right), \\ \delta\theta_{12} &\approx \frac{1}{\sqrt{2}} \left(\theta_{12}^e e^{-i\delta_{12}^e} + \theta_{13}^e e^{-i\delta_{13}^e} \right). \end{aligned} \quad (\text{C.7})$$

Because of the phases, these corrections $\delta\theta_{ij}$ are in general complex. Our search assumes no phases in both \mathcal{V} and U_{seesaw} , so that the $\delta\theta_{ij}$ are approximately real,⁸ and serve as corrections to the seesaw mixing angles.

The required corrections to the seesaw mixing angles can be computed from their global fits [12]. For the TBM mixing ($\theta_{12}^\nu \approx 35.3^\circ$) and using the 3σ range of neutrino mixing angles, the range of the required corrections are

$$\delta\theta_{23} \sim (-9^\circ \leftrightarrow 8^\circ), \quad |\delta\theta_{13}| \sim (8^\circ \leftrightarrow 10^\circ), \quad \delta\theta_{12}^{TBM} \sim (-4.8^\circ \leftrightarrow 1.7^\circ) \quad (\text{C.8})$$

while for BM mixing ($\theta_{12}^\nu = 45^\circ$) the required $\delta\theta_{12}$ is instead

$$\delta\theta_{12}^{BM} \sim (8^\circ, 14.5^\circ), \quad (\text{C.9})$$

much larger than that for TBM mixing.

These corrections can be further translated into the mixing angles θ_{ij}^e in \mathcal{U}_{-1} by using Eq.(C.7). Taking into account possible negative signs generated by the phases δ_{ij}^e , the magnitude range of θ_{23}^e is given by

$$|\theta_{23}^e| \sim (0^\circ \leftrightarrow 9^\circ), \quad (\text{C.10})$$

and the allowed regions for the magnitudes of θ_{12}^e and θ_{13}^e are given in Fig.6.

In Fig.6, one notices that the regions for the TBM and BM mixings are disconnected. For the TBM case, the magnitudes of θ_{12}^e and θ_{13}^e tend to be close to each other, while they are quite apart for the BM case. This feature is indeed confirmed by the two-angle solutions, as can be seen in Fig.7.

⁸Small non-zero phases can be generated for two reasons: the CKM matrix is only unitary to the order of λ^3 ; the assignment of GJ factors.

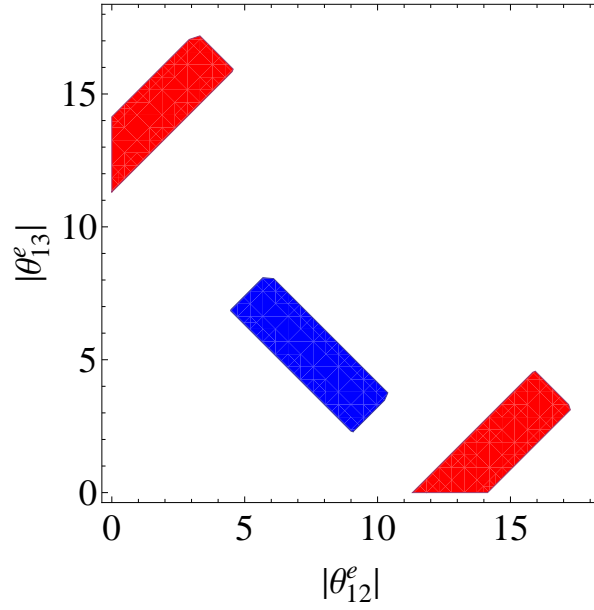


Figure 6: Allowed regions for the magnitudes of θ_{12}^e and θ_{13}^e . The blue and red regions correspond to the TBM and BM mixings respectively.

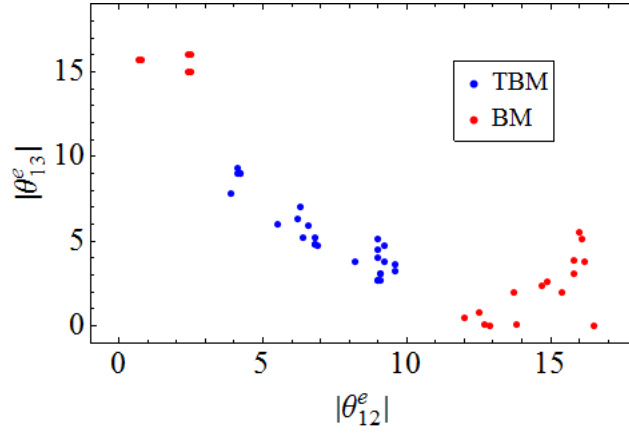


Figure 7: A scatter plot of the magnitudes of θ_{12}^e and θ_{13}^e in the two-angle solutions.

- $|s'_{13}|, |s_{12}^e|, |s_{13}^e|, |c_{23}^e| \ll 1$:

In this case, at the leading order the MNSP mixing angles can be written as

$$\begin{aligned}
s_{23}e^{-i\delta_{23}} &\approx -c_{23}^\nu e^{-i\delta_{23}^e} + s_{23}^\nu c_{23}^e e^{-i\delta_{23}^\nu}, \\
\theta_{13}e^{-i\delta_{13}} &\approx \theta_{13}^\nu e^{-i\delta_{13}^\nu} + c_{23}^\nu \theta_{12}^e e^{-i(\delta_{12}^e + \delta_{23}^e)} - s_{23}^\nu \theta_{13}^e e^{-i(\delta_{23}^\nu + \delta_{13}^e - \delta_{23}^e)}, \\
s_{12}e^{-i\delta_{12}} &\approx s_{12}^\nu e^{-i\delta_{12}^\nu} - c_{12}^\nu s_{23}^\nu \theta_{12}^e e^{i(\delta_{23}^\nu - \delta_{12}^e - \delta_{23}^e)} - c_{12}^\nu c_{23}^\nu \theta_{13}^e e^{-i(\delta_{13}^e - \delta_{23}^e)},
\end{aligned} \tag{C.11}$$

Applying the above general formulae to the previous form of $\mathcal{U}_{\text{seesaw}}$, one arrives at

$$\begin{aligned}
-s_{23}e^{-i(\delta_{23} - \delta_{23}^e)} &= \sin\left(\frac{\pi}{4} + \delta\theta_{23}\right), \\
\theta_{13}e^{-i\delta_{13}} &= \delta\theta_{13}, \\
-s_{12}e^{-i\delta_{12}} &= \sin(\theta_{12}^\nu + \delta\theta_{12}),
\end{aligned} \tag{C.12}$$

where

$$\begin{aligned}
\delta\theta_{23} &\approx c_{23}^e e^{i\delta_{23}^e}, \\
\delta\theta_{13} &\approx \frac{1}{\sqrt{2}} \left(\theta_{12}^e e^{-i(\delta_{12}^e + \delta_{23}^e)} - \theta_{13}^e e^{-i(\delta_{13}^e - \delta_{23}^e)} \right), \\
\delta\theta_{12} &\approx -\frac{1}{\sqrt{2}} \left(\theta_{12}^e e^{-i(\delta_{12}^e + \delta_{23}^e)} + \theta_{13}^e e^{-i(\delta_{13}^e - \delta_{23}^e)} \right).
\end{aligned} \tag{C.13}$$

The structure of the above corrections is quite similar to the previous case, except for different phases and the replacement of θ_{23}^e by c_{23}^e . As these phases are not relevant to our search, one can perform a similar analysis on the θ_{ij}^e , and eventually we find the above formulae agree with the two-angle results quite well.

Having established the required mixing angles in \mathcal{U}_{-1} , can they be obtained from $Y^{(-1/3)}$ by using $SU(5)$?

From $Y^{(-1/3)}$ to $Y^{(-1)}$

In the search, $Y^{(-1)}$ is obtained by first transposing $Y^{(-1/3)}$ with GJ factors in some entries. As a result the mixing angles β_{ij} in \mathcal{V} can differ from those θ_{ij}^e in \mathcal{U}_{-1} . Their relations will be the focus of this section. The mass ratios of the charged leptons are also affected by the GJ factors, but they are harder to investigate analytically.

Although in the solutions the β_{ij} can appear in all four quadrants, we consider an example where they are all in the first or fourth quadrant and close to the x-axis. The other cases can be studied similarly with possibly different signs and $\sin\beta_{ij}$ replaced by $\cos\beta_{ij}$.

\mathcal{V} is parametrized as

GJ entries	θ_{12}^e	θ_{13}^e
(22)	$\beta_{12}/3$	β_{13}
(21), (22)	$4A\beta_{13} \pm \beta_{12}$	β_{13}
(22), (23)	$4A\beta_{13} \pm \beta_{12}/3$	β_{13}
(22), (31)	$4A\beta_{13} \pm \beta_{12}/3$	$3\beta_{13}$
(21), (22), (23)	β_{12}	β_{13}
(21), (22), (31)	β_{12}	$3\beta_{13}$
(22), (23), (31)	$8A\beta_{13} \pm \beta_{12}/3$	$3\beta_{13}$
(21), (22), (23), (31)	$12A\beta_{13} \pm \beta_{12}$	$3\beta_{13}$

Table 17: Relations between (β_{12}, β_{13}) and $(\theta_{12}^e, \theta_{13}^e)$ under the influences of GJ factors when β_{23} is small.

$$\mathcal{V} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta_{23} \\ 0 & -\beta_{23} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \beta_{13} \\ 0 & 1 & 0 \\ -\beta_{13} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta_{12} & 0 \\ -\beta_{12} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{C.14})$$

resulting in the transposed $Y^{(-1/3)}$,

$$Y^{(-1/3)T} \sim \begin{pmatrix} \times & \beta_{13}A\lambda^2 + \beta_{12}\lambda^2/3 & \beta_{13} \\ \times & \lambda^2/3 & -A\lambda^4/3 + \beta_{23} \\ \times & A\lambda^2 & 1 \end{pmatrix}, \quad (\text{C.15})$$

where high order corrections are neglected, and the entries in the first column are not shown, as they have no impact on obtaining the mixing angle θ_{ij}^e .

Although any entries in the above matrix can be assigned GJ factors, not all of them have significant impacts on θ_{ij}^e . Next we will first identify what the relevant GJ entries are, and then evaluate their consequences.

Since the (23) entry in this matrix is small, singling out θ_{23}^e will not affect the other two angles by much, and vice versa. Therefore the relevant GJ entry for θ_{23}^e would be the (32) entry of $Y^{(-1/3)}$, and its effect is to have θ_{23}^e nearly three times larger than β_{23} .

The role of the other two mixing angles θ_{12}^e and θ_{13}^e is more complicated because of the relatively large (32) entry, $A\lambda^2$. In this case the important GJ entries are in the (21), (22), (23) and (31) elements of $Y^{(-1/3)}$. Since the (22) entry needs to be multiplied in order to obtain a correct value of m_μ , we are left with eight different ways of assigning GJ factors to the remaining three entries. Among these eight cases we choose one, where GJ factors appear only in the (22), (23) and (31) entries, as an example to show how the relations between (β_{12}, β_{13}) and $(\theta_{12}^e, \theta_{13}^e)$ are derived, and then list the results for the other cases in Table 17.

In this case, $Y^{(-1)}$ looks like

$$Y^{(-1)} \sim \begin{pmatrix} \times & \beta_{13}A\lambda^2 + \beta_{12}\lambda^2/3 & -3\beta_{13} \\ \times & -\lambda^2 & \times \\ \times & -3A\lambda^2 & 1 \end{pmatrix}. \quad (\text{C.16})$$

While $\theta_{13}^e = -3\beta_{13}$ is determined by the (13) entry in Eq.(C.16), θ_{12}^e is affected by the relatively large (12) entry, yielding,

$$\theta_{12}^e = -(8A\beta_{13} + \beta_{12}/3). \quad (\text{C.17})$$

Note that in Table 17 we also include the cases where β_{12} and β_{13} are in the second or third quadrant. These possible choices of quadrants also lead to the possibilities of the “ \pm ” signs in the above table. Overall negative signs are omitted in the sense that all β_{ij} and θ_{ij} should be interpreted as deviations from the axes. These formulae agree with the two-angle results quite well.

The formulae in the above table can be used to find out what the allowed parameter space of the input mixing angle β_{ij} is, as the required θ_{ij}^e have been found previously. For β_{23} , Eq.(C.10) indicates that it is at most 10° away from the axes for both the TBM and BM cases, while the allowed parameter space of the other two angles has to be studied separately for the TBM and BM cases, as we notice that in Fig.6 their parameter space has no overlap.

With the help of Fig.6 and the formulae in Table 17, one then finds that β_{13} is at most 10° (20°) away from the axes for the TBM(BM) case, while β_{12} can take on all possible values if one takes into account possible cancellations between the terms $4A\beta_{13}$ and $\beta_{12}/3$ in Table 17. These results have been used to guide our three-angle search.

As a final remark, the formulae in Table 17 may also shed some light on understanding the null search result for a symmetric $Y^{(-1/3)}$. Given a symmetric $Y^{(-1/3)}$, β_{13} and β_{12} would be around 0.2° and 13° respectively. This immediately eliminates the TBM case, as θ_{13}^e is now too small, while for the BM case, one is still left with several possible assignments of GJ factors. In this case it may be that all such assignments are incompatible with the mass ratios of charged leptons, and are also ruled out eventually.

D Subtleties

In this appendix we address several issues concerning the quadrants of the angles in $\mathcal{U}_{\text{MNSP}}$ and $\mathcal{U}_{\text{seesaw}}$. Adopting the parametrization of Eq.(9) for \mathcal{V} to decompose $\mathcal{U}_{\text{MNSP}}$ and $\mathcal{U}_{\text{seesaw}}$, we find that all of the TBM angles lie in the fourth quadrant, up to an ambiguity for θ_{13}^ν , which is zero. As experiments only measure the \sin^2 or \cos^2 of the MNSP mixing angles, the quadrants in which they lie are not determined.

The alert reader may wonder whether this opens a back door for solutions with large mixing angles in \mathcal{U}_{-1} ; having large mixing angles in \mathcal{U}_{-1} would contradict the conventional wisdom that the seesaw matrix accounts for large mixing

angles in $\mathcal{U}_{\text{MNSP}}$. In a simple numerical study (see later), we indeed find that large mixing angles in \mathcal{U}_{-1} can occur if one allows for such an ambiguity in $\mathcal{U}_{\text{MNSP}}$. Fortunately, because in the search we restrict some of the input angles in \mathcal{V} to be close to the x-axis, we find the cases with large angles in \mathcal{U}_{-1} do not occur.

Another quadrant issue that one may worry about in the search is whether the quadrants of $\mathcal{U}_{\text{MNSP}}$ and $\mathcal{U}_{\text{seesaw}}$ match. This would be expected if the seesaw matrix indeed accounts for the large mixing angles and \mathcal{U}_{-1} provides only small corrections. Although in the search we do not require such a match, we will show that such an issue actually does not exist, due to the restriction to small mixing angles in \mathcal{U}_{-1} and an ambiguity in relocating phases when diagonalizing $Y^{(-1)}$.

Finally, we highlight the freedom available in choosing the quadrants of the angles in $\mathcal{U}_{\text{seesaw}}$, and show that transferring to a different set of quadrants does not affect the search results, up to a change in the quadrants of the input angles.

Large mixing angles in \mathcal{U}_{-1} ?

We begin by discussing the issue of large mixing angles in \mathcal{U}_{-1} though a numerical example given in Table 18. Fixing the seesaw mixing matrix $\mathcal{U}_{\text{seesaw}}$ to be the TBM matrix in Eq.(10), we allow for the mixing angles in $\mathcal{U}_{\text{MNSP}}$ to be in any quadrant.

The mixing angles in \mathcal{U}_{-1} can then be obtained through

$$\mathcal{U}_{-1} = \mathcal{U}_{\text{seesaw}} \mathcal{U}_{\text{MNSP}}^\dagger, \quad (\text{D.1})$$

assuming the best fit values for $\mathcal{U}_{\text{MNSP}}$, and fall into two categories according to the choice for the quadrants of $\mathcal{U}_{\text{MNSP}}$. Half lead to three large angles, while the other half give small angles.

Quadrants of $(\theta_{12}, \theta_{13}, \theta_{23})$ in $\mathcal{U}_{\text{MNSP}}$	θ_{12}^e	θ_{13}^e	θ_{23}^e
111, 113, 132, 134, 321, 323, 342, 344	64.5°	34.5°	69.6°
112, 114, 131, 133, 322, 324, 341, 343	62.2°	40.6°	35.5°
121, 123, 142, 144, 311, 313, 332, 334	64.5°	34.5°	20.4°
122, 124, 141, 143, 312, 314, 331, 333	62.2°	40.6°	54.5°
211, 213, 232, 234, 421, 423, 442, 444	7.0°	5.6°	3.8°
212, 214, 231, 233, 422, 424, 441, 443	4.6°	7.7°	85.9°
221, 223, 242, 244, 411, 413, 432, 434	7.0°	5.6°	86.2°
222, 224, 241, 243, 412, 414, 431, 433	4.6°	7.7°	4.1°

Table 18: Mixing angles in \mathcal{U}_{-1} for a given choice of quadrants of the mixing angles in $\mathcal{U}_{\text{MNSP}}$. Best fit values in Table 1 are used for θ_{ij} , $\mathcal{U}_{\text{seesaw}} = \mathcal{U}_{\text{TBM}}$, and θ_{ij}^e are interpreted as deviations from the x-axis.

Since this occurrence of large angles in \mathcal{U}_{-1} would destroy the idea of using the seesaw matrix to explain large angles in $\mathcal{U}_{\text{MNSP}}$, we therefore restrict our-

selves only to the cases where the mixing angles in \mathcal{U}_{-1} are small, or close to 90° .⁹

In Table 18, one also notices that when the quadrants of $\mathcal{U}_{\text{MNSP}}$, say (444) or (414), match with those of $\mathcal{U}_{\text{seesaw}}$, the mixing angles in \mathcal{U}_{-1} are small. However, there also exist other choices that give rise to small angles in \mathcal{U}_{-1} but are not matched to the quadrants of $\mathcal{U}_{\text{seesaw}}$. Since in the search we have included these choices as well, one may wonder whether we still have such a matching requirement. Next we will try to argue that that requirement is still present, although we do not impose it explicitly.

Matching the quadrants of $\mathcal{U}_{\text{MNSP}}$ and $\mathcal{U}_{\text{seesaw}}$

We begin by invoking the Iwasawa decompositions of \mathcal{U}_{-1} and $\mathcal{U}_{\text{seesaw}}$ in Eq.(C.1). Although in our current search phases are not included in \mathcal{V} and $\mathcal{U}_{\text{seesaw}}$, the phase matrices P_L and P_R can still occur if we have fixed a particular convention for the quadrants of angles. In other words, these phase matrices contain the information about the quadrants of the angles.

We then obtain the MNSP matrix through

$$\begin{aligned}\mathcal{U}_{\text{MNSP}} &= \mathcal{U}_{-1}^\dagger \mathcal{U}_{\text{seesaw}} \\ &= P_R^{e\dagger} R_{12}^{e\dagger} U_{13}^{e\dagger} R_{23}^{e\dagger} P_L^{e\dagger} P_L^\nu R_{23}^\nu U_{13}^{\nu\nu} R_{12}^\nu P_R^\nu \\ &= (P_R^{e\dagger} P_L^{e\dagger} P_L^\nu) U_{12}^{e\dagger} U_{13}^{e\dagger} U_{23}^{e\dagger} R_{23}^\nu U_{13}^{\nu\nu} R_{12}^\nu P_R^\nu,\end{aligned}\quad (\text{D.2})$$

where instead of commuting all the phase matrices to the left as we did in Eq.(C.2), we now only commute $P_L^{e\dagger} P_L^\nu$ to the left, with the phases in U_{ij}^e given by

$$\begin{aligned}\delta_{12}^e &= (\varphi_2^\nu - \varphi_2^e) - (\varphi_1^\nu - \varphi_1^e), \\ \delta_{13}^e &= \delta_{13}^{e'} - (\varphi_3^\nu - \varphi_3^e) + (\varphi_1^\nu - \varphi_1^e), \\ \delta_{23}^e &= (\varphi_3^\nu - \varphi_3^e) - (\varphi_2^\nu - \varphi_2^e).\end{aligned}$$

Matching the quadrants of the angles in $\mathcal{U}_{\text{seesaw}}$ and $\mathcal{U}_{\text{MNSP}}$ can then be discussed based on the last line of Eq.(D.2). First, because in the search we restrict ourselves to the case where the mixing angles in \mathcal{U}_{-1} are small, multiplying $U_{12}^{e\dagger} U_{13}^{e\dagger} U_{23}^{e\dagger}$ by $R_{23}^\nu U_{13}^{\nu\nu} R_{12}^\nu$ will not change the quadrants of the mixing angles in $R_{23}^\nu U_{13}^{\nu\nu} R_{12}^\nu$. We are then left to match the left and right diagonal phase matrices between $\mathcal{U}_{\text{seesaw}}$ and $\mathcal{U}_{\text{MNSP}}$. From Eq.(C.1), one can see that the right phase matrix P_R^ν has already been matched, while for the left phase matrix we must require

$$P_R^{e\dagger} P_L^{e\dagger} P_L^\nu \approx P_L^\nu, \quad (\text{D.3})$$

⁹The reason that we also study the latter case is because we think having θ_{23}^e close to 90° seems special and may have implications for model building.

which implies that in the decomposition of \mathcal{U}_{-1} , the phases in P_R^e have to cancel those in P_L^e . For a generic \mathcal{U}_{-1} , this certainly may not hold. However, in the diagonalizaion of $Y^{(-1)}$,

$$\begin{aligned} Y^{(-1)} &= \mathcal{U}_{-1} \mathcal{D}_{-1} \mathcal{V}_{-1}^\dagger, \\ &= (P_L^e R_{23}^e U_{13}^e R_{12}^e P_R^e) \mathcal{D}_{-1} \mathcal{V}_{-1}^\dagger, \end{aligned} \quad (\text{D.4})$$

the ambiguity of relocating phases among P_R^e , \mathcal{D}_{-1} , and \mathcal{V}_{-1} allows one to *choose* the phases in P_R^e to cancel those in P_L^e , eventually resulting in a match of the quadrants between $\mathcal{U}_{\text{seesaw}}$ and $\mathcal{U}_{\text{MNSP}}$. We therefore do not have to impose such an extra matching criterion in our current search, although one may have to include it if one allows for large mixing angles in \mathcal{U}_{-1} .

Quadrants of the angles in $\mathcal{U}_{\text{seesaw}}$

As a final point, we discuss the freedom available in choosing the quadrants of the angles in $\mathcal{U}_{\text{seesaw}}$. In the main text a particular form of $\mathcal{U}_{\text{seesaw}}$ is chosen, where the mixing angles θ_{12}' and θ_{13}' are both in the fourth quadrant according to the parametrization used in Eq.(9). Although we made this choice in order to be consistent with our previous publication [5], one can certainly choose a different form of $\mathcal{U}_{\text{seesaw}}$ by setting its angles in different quadrants. The search results will be altered by doing so, however, we find that the only change is in the quadrants of the input angles β_{ij} ; their actual deviations from the x-axis stay the same as before. Next we will give an example to illustrate this.

Suppose that we were to instead set both θ_{12}' and θ_{13}' in $\mathcal{U}_{\text{seesaw}}$ to be in the first quadrant. A transformation can then be found between this new seesaw matrix and the previous one,

$$\mathcal{U}'_{\text{seesaw}} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \mathcal{U}_{\text{seesaw}} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad (\text{D.5})$$

where a superscript “'” is used to distinguish these two cases.

With this new $\mathcal{U}'_{\text{seesaw}}$, the MNSP matrix now becomes

$$\mathcal{U}'_{\text{MNSP}} = \mathcal{U}_{-1}^\dagger \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \mathcal{U}_{\text{seesaw}} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad (\text{D.6})$$

which implies that if one could also relate the above new \mathcal{U}'_{-1} with \mathcal{U}_{-1} through

$$\mathcal{U}'_{-1} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \mathcal{U}_{-1}, \quad (\text{D.7})$$

the new MNSP matrix would have the same mixing angles as the previous case, up to some unknown Majorana phases. Since in our search $Y^{(-1)}$ is related to $Y^{(-1/3)}$ through $SU(5)$, such a relation between \mathcal{U}'_{-1} and \mathcal{U}_{-1} can indeed be realized by performing a transformation on the input matrix \mathcal{V}

$$\mathcal{V}' = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \mathcal{V}, \quad (\text{D.8})$$

which gives

$$\begin{aligned} \beta'_{23} &= -\beta_{23}, \\ \beta'_{13} &= \pi + \beta_{13}, \\ \beta'_{12} &= \beta_{12}. \end{aligned} \quad (\text{D.9})$$

Other cases with different choices of quadrants can be studied similarly, and the same conclusion holds. Therefore in the search one can make an arbitrary choice for the quadrants of the angles in $\mathcal{U}_{\text{seesaw}}$, as a particular choice may be related to any other by a change of the quadrants of the input angles.

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